Algorithmic Operations Research

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Volume 1, Number 1, Summer 2006

URI: https://id.erudit.org/iderudit/aor1_1art05

See table of contents

Publisher(s)

Preeminent Academic Facets Inc.

ISSN

1718-3235 (digital)

Explore this journal

Cite this article

Gutin, G., Koller, A. & Yeo, A. (2006). Note on Upper Bounds for TSP Domination Number. *Algorithmic Operations Research*, *1*(1), 52–54.

Article abstract

The domination number, domn(A, n), of a heuristic A for the Asymmetric TSP is the maximum integer d = d(n) such that, for every instance I of the Asymmetric TSP on n cities, A produces a tour T which is not worse than at least d tours in I including T itself. Two upper bounds on the domination number are proved.

Algorithmic Operations Research Vol.1 (2006) 52–54

Note on Upper Bounds for TSP Domination Number

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Abstract

The domination number, $\mathrm{domn}(\mathcal{A}, n)$*, of a heuristic* $\mathcal{A}$ *for the Asymmetric TSP is the maximum integer* $d = d(n)$ *such that, for every instance* $\mathcal{I}$ *of the Asymmetric TSP on* n *cities,* $\mathcal{A}$ *produces a tour* T *which is not worse than at least* d *tours in* $\mathcal{I}$ *including* T *itself. Two upper bounds on the domination number are proved.*

Key words: Traveling salesman problem, domination analysis, complexity, approximation algorithms

1. Introduction

The *Asymmetric Traveling Salesman Problem (ATSP)* is stated as follows. Given a weighted complete digraph (K_n^*, w), find a Hamilton cycle (called a *tour*) in K_n^* of minimum cost. Here the *weight function* w is a mapping from $A(K_n^*)$, the set of arcs in K_n^*, to the set of reals. The *weight* of an arc xy of K_n^* is $w(x, y)$. The *weight* $w(D)$ of a subdigraph D of K_n^* is the sum of the weights of arcs in D.

It is well known that most combinatorial optimization problems including the ATSP are $\mathcal{NP}$-hard. Due to the lack of polynomial time algorithms to solve $\mathcal{NP}$-hard problems to optimality, researchers and practitioners often use various heuristics such as local search and genetic algorithms that usually provide good solutions for instances that arise in practice. Very often heuristics do not have any theoretical guarantee for the optimization problem under consideration, i.e., for some instances of the problem the value of heuristic solution is arbitrary far from the optimum. Hence, normally various heuristics for the same problem are compared in computational experiments. The outcomes of computational experiments heavily rely on the authors choice of families of instances and, thus, are non-objective.

With this state of affairs in mind, Glover and Punnen [3] suggested a new approach for evaluation of heuristics that compares heuristics according to their so-called domination ratio. We define this notion only for the ATSP since its extension to other problems is obvious. The *domination number*, $\mathrm{domn}(\mathcal{A}, n)$, of a heuristic $\mathcal{A}$ for the ATSP is the maximum integer $d = d(n)$ such that, for every instance $\mathcal{I}$ of the ATSP on n cities, $\mathcal{A}$ produces a tour T which is not worse than at least d tours in $\mathcal{I}$ including T itself. The ratio $\mathrm{domr}(\mathcal{A}, n) = \mathrm{domn}(\mathcal{A}, n)/(n-1)!$, i.e., the domination number divided by the total number of tours, is the *domination ratio* of $\mathcal{A}$.

It is known the nearest neighbor algorithm for the ATSP is of domination number 1 (first proved in [7]). This means that for every $n \geq 2$, there is an instance of ATSP on n vertices, for which the nearest neighbor algorithm finds the *unique* worst possible tour. Since the number of distinct tours in an n-vertex complete digraph is $(n-1)!$, we see that the nearest neighbor algorithm is of domination ratio $1/(n-1)!$. There are many ATSP algorithms of domination number at least $(n-2)!$ [8], i.e., in the worst case they guarantee that their tour is at least as good as $(n-2)! - 1$ *other* tours.

Clearly, the domination ratio is well defined for every heuristic and, for the same optimization problem, a heuristic with higher domination ratio may be considered a better choice than a heuristic with lower domination ratio. Ben-Arieh et al. [1] compared two heuristics for the generalized ATSP. The heuristics performed equally well in computational experiments, but it was proved that one of them has a significantly larger domination number. For the Symmetric TSP, Punnen, Margot and Kabadi [9] showed that after a polynomial number of iterations the domination number of the best improvement 2-Opt that uses small neighborhoods significantly exceeds that of the best improvement local search based on neighborhoods of much larger cardinality. Punnen, Margot and Kabadi [9] and other papers have led Gutin and Yeo [6] to the conclusion that the cardinality of the neighborhood used by a local search is not the right

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measure of the effectiveness of the local search. Domination ratio, along with some other parameters such as the diameter of the neighborhood digraph (see Gutin, Yeo and Zverovitch [8]), provide a much better measure.

Already Glover and Punnen [3] were interested in the possible range of the domination number of *polynomial time* ATSP heuristics. There are several papers devoted to lower bounds on the maximum domination number of such heuristics, for references see [8]. In this note we concentrate on upper bounds.

In [5], upper bounds were obtained for the cardinality of polynomial time searchable ATSP neighborhoods. In this note we show that minor changes in the proofs of [5] lead us to stronger and more general results on the maximum possible domination number of ATSP heuristics when the running time is restricted. We also prove a result, Theorem 2..4, that gives an absolute upper bound of the domination number of polynomial time ATSP heuristics. Theorem 2..4 improves Theorem 20 in [9]. Our proof is a modification of the proof of Theorem 20 in [9].

2. Upper Bounds

It is realistic to assume that any ATSP algorithm spends at least one unit of time on every arc of the complete digraph K_n^* that it considers. We use this assumption in the rest of this section.

Theorem 2..1 *Let $\mathcal{A}$ be an ATSP heuristic that runs in time at most $t(n)$. Then the domination number of $\mathcal{A}$ does not exceed $\max_{1\le n'\le n}(t(n)/n')^{n'}$.*

PROOF. Let $D = (K_n^*, w)$ be an instance of ATSP and let H be the tour that $\mathcal{A}$ returns, when its input is D. Let $DOM(H)$ denote all tours in D which are not lighter than H including H itself. We assume that D is a worst instance for $\mathcal{A}$, namely $\mathrm{domn}(\mathcal{A}, n) = |DOM(H)|$. Since $\mathcal{A}$ is arbitrary, to prove this theorem, it suffices to show that $|DOM(H)| \le \max_{1\le n'\le n}(t(n)/n')^{n'}$.

Let E denote the set of arcs in D, which $\mathcal{A}$ actually examines; observe that $|E| \le t(n)$ by the assumption above. Let $A(H)$ be the set of arcs in H. Let F be the set of arcs in H that are not examined by $\mathcal{A}$, and let G denote the set of arcs in $D - A(H)$ that are not examined by $\mathcal{A}$.

We first prove that every arc in F must belong to each tour of $DOM(H)$. Assume that there is a tour $H' \in DOM(H)$ that avoids an arc $a \in F$. If we assign to a a very large weight, H' becomes lighter than H, a contradiction.

Similarly, we prove that no arc in G can belong to a tour in $DOM(H)$. Assume that $a \in G$ and a is in a tour $H' \in DOM(H)$. By making a very light (possibly negative), we can ensure that $w(H') < w(H)$, a contradiction.

Now let D' be the digraph obtained by contracting the arcs in F and deleting the arcs in G, and let n' be the number of vertices in D'. Note that every tour in $DOM(H)$ corresponds to a tour in D' and, thus, the number of tours in D' is an upper bound on $|DOM(H)|$. In a tour of D', there are at most $d^+(i)$ possibilities for the successor of a vertex i, where $d^+(i)$ is the out-degree of i in D'. Hence we obtain that

$$|DOM(H)| \le \prod_{i=1}^{n'} d^+(i) \le \left(\frac{1}{n'}\sum_{i=1}^{n'} d^+(i)\right)^{n'} \le \left(\frac{t(n)}{n'}\right)^{n'},$$

where we applied the arithmetic-geometric mean inequality. □

Corollary 2..2 *Let $\mathcal{A}$ be an ATSP heuristic that runs in time at most $t(n)$. Then the domination number of $\mathcal{A}$ does not exceed $\max\{e^{t(n)/e}, (t(n)/n)^n\}$, where e is the basis of natural logarithms.*

PROOF. Let $U(n) = \max_{1\le n'\le n}(t(n)/n')^{n'}$. By differentiating $f(n') = (t(n)/n')^{n'}$ with respect to n' we can readily obtain that $f(n')$ increases for $1 \le n' \le t(n)/e$, and decreases for $t(n)/e \le n' \le n$. Thus, if $n \le t(n)/e$, then $f(n')$ increases for every value of $n' < n$ and $U(n) = f(n) = (t(n)/n)^n$. On the other hand, if $n \ge t(n)/e$ then the maximum of $f(n')$ is for $n' = t(n)/e$ and, hence, $U(n) = e^{t(n)/e}$. □

The next assertion follows directly from the proof of Corollary 2..2.

Corollary 2..3 *Let $\mathcal{A}$ be an ATSP heuristic that runs in time at most $t(n)$. For $t(n) \ge en$, the domination number of $\mathcal{A}$ does not exceed $(t(n)/n)^n$.*

Note that the restriction $t(n) \ge en$ is important since otherwise the bound of Corollary 2..3 can be invalid. Indeed, if $t(n)$ is a constant, then for n large enough the upper bound becomes smaller than 1, which is not correct since the domination number is always at least 1.

It is proved in [4] that there are $O(n)$-time ATSP algorithms of domination number $2^{\Theta(n)}$. It follows from the last corollary that this result cannot be improved.

Theorem 2..4 *Unless P=NP, there is no polynomial time ATSP algorithm of domination number at least* $(n-1)! - \lfloor n - n^{\alpha} \rfloor!$ *for any constant* $\alpha < 1$.

PROOF. Assume that there is a polynomial time algorithm $\mathcal{H}$ with domination number at least $(n-1)! - \lfloor n - n^{\alpha} \rfloor!$ for some constant $\alpha < 1$. Choose an integer $s > 1$ such that $\frac{1}{s} < \alpha$.

Consider a weighted complete digraph (K_n^*, w). We may assume that all weights are non-negative as otherwise we may add a large number to each weight. Choose any pair of distinct vertices u and v in K_n^*. Consider another complete digraph D on $n^s - n$ vertices, in which all weights are 0 and which is vertex disjoint from K_n^*. Add all possible arcs between K_n^* and D such that the weights of all arcs coming into u and going out of v are 0 and the weights of all other arcs are M, where M is larger than n times the maximum weight in (K_n^*, w). Let the resulting weighted complete digraph be denoted by $K_{n^s}^*$ and note that we have now obtained an instance $(K_{n^s}^*, w')$ of ATSP.

Apply $\mathcal{H}$ to $(K_{n^s}^*, w')$ (observe that $\mathcal{H}$ is polynomial in n for $(K_{n^s}^*, w')$). Notice that there are exactly $(n^s - n)!$ Hamilton cycles in $(K_{n^s}^*, w')$ of weight L, where L is the weight of a lightest Hamilton (u, v)-path in K_n^*. Each of the $(n^s - n)!$ Hamilton cycles is obviously optimal. Observe that the domination number of $\mathcal{H}$ on $K_{n^s}^*$ is at least $(n^s - 1)! - \lfloor n^s - (n^s)^{\alpha} \rfloor!$. However, for sufficiently large n, we have

$$(n^s - 1)! - \lfloor n^s - (n^s)^{\alpha} \rfloor! \geq (n^s - 1)! - (n^s - n)! + 1$$

as $n^{s\alpha} \geq n + 1$ for n large enough. Thus, a Hamilton cycle produced by $\mathcal{H}$ is always among the optimal solutions (for n large enough). This means that we can obtain a lightest Hamilton (u, v)-path in K_n^* in polynomial time, which is impossible since the lightest Hamilton (u, v)-path problem is a well-known NP-hard problem. We have arrived at a contradiction. □

Received 22 Dec 2004; revised 25 April 2005; accepted 10 July 2005

Acknowledgement This research was partially supported by a Leverhulme Foundation grant. Research of Gutin was supported in part by the IST Programme of the European Community, under the PASCAL Network of Excellence, IST-2002-506778.

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