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Article abstract

Even if no explicit meta-discourse on mathematics is found in pre-modern China outside of mathematical writings, reflections upon objects of mathematical inquiry and the mathematician's toolbox existed. They are shown to be built into the corpus of Chinese mathematics itself. As illustrated in particular through one specific mathematical domain that evolved from the first to the 19th century, such philosophical reflections are dispersed between texts, paratexts and images, thereby borrowing concepts and iconic images from other Chinese contexts of philosophical nature.

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NUMBERS AND NARRATIVES — OR : WHEN RUSSELL MEETS ZHU SHIJIE TO DISCUSS PHILOSOPHY OF MATHEMATICS

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RÉSUMÉ : Même si aucune source métadiscursive sur les mathématiques elles-mêmes n'a été transmise de la Chine ancienne et prémoderne, des réflexions ont été menées sur les objets mathématiques et sur la boîte à outils des praticiens. Cet article montre comment elles sont insérées dans les traités mêmes en prenant en particulier l'exemple d'un domaine de la théorie des nombres qui a évolué en Chine du premier à la fin du XIXe siècle. Ces réflexions sont dispersées entre textes, paratextes et diagrammes tout en empruntant des concepts et des figures iconiques d'autres contextes de nature philosophique.

ABSTRACT : Even if no explicit meta-discourse on mathematics is found in pre-modern China outside of mathematical writings, reflections upon objects of mathematical inquiry and the mathematician's toolbox existed. They are shown to be built into the corpus of Chinese mathematics itself. As illustrated in particular through one specific mathematical domain that evolved from the first to the 19th century, such philosophical reflections are dispersed between texts, paratexts and images, thereby borrowing concepts and iconic images from other Chinese contexts of philosophical nature.

I. *INTERVISTA IMPOSSIBILE*[1]

Bertrand Russell : I have just finished writing my *Introduction to Mathematical Philosophy* while I was in prison. In the process of preparing a trip to China to give some lectures on mathematical logic,[2] I would like to learn more about the philosophy of mathematics in China from you, as a practitioner. Let us begin with a simple question inspired by Frege's *Grundlagen der Arithmetik* : how would you define numbers ?

Zhu Shijie : I wouldn't do that, it's useless in calculations.

1. *Le interviste impossibili* is the title of a radio program of RAI 2 from the years 1974-1975 in which real contemporary personalities pretend to be interviewing people from another era.
2. On Russell's lectures in China and Chinese translations of his work, cf. XU Yibao, "Bertrand Russell and the Introduction of Mathematical Logic in China", *History and Philosophy of Logic*, 24 (2003), p. 181-196.

Russell : All right, I see. Maybe I shall reformulate my question, starting from the simple. Since I assume you have a certain concept in mind of what a number is when you do mathematics, what about unity ?

Zhu : The *Classic of Changes* (*Yijing* 易經) says it is heaven, the Supreme Ultimate (*taiji* 太極).

Russell : I am confident that as a literati you know the Confucian Classics by heart very well, but I wanted to learn more about the philosophy of mathematics.

Zhu : May I ask what you mean by "philosophy of mathematics" ?

Russell : No embarrassing questions,[3] please, I am the one who shall ask them. Let us rather proceed further with the interview. So, according to your quotation from the *Changes*, unity is something foundational, but how about the other numbers ?

Zhu : There are no other numbers, unity is the only one, ten thousand things originate from it.[4]

Russell : If I understand you correctly, beyond unity you consider integers only, that is, a series of natural numbers from one to ten thousand obtained by adding "1" repeatedly ?

Zhu : From one make two, from two make four, from four make eight, the process of generation never ends. How could this not be of itself naturally so ?

Russell : If I am not mistaken, this procedure does not produce numbers other than powers of two, thus not even ten thousand ?

Zhu : Ten thousand are so many that we cannot even count them.

Russell : A bit confusing…, so your "naturally so" numbers are an uncountable finite set ?

Zhu : If you don't grasp my meaning, dear Sir Russell, I suggest you look at the River Chart (*Hetu* 河圖) and the Inscription of the River Luo (*Luoshu* 洛書), the principles will immediately become apparent.

Russell : But these diagrams only show numbers from one to ten, how should they convey a general philosophical meaning ? Maybe you could elaborate on the kind of mathematical principles you see in these diagrams, such as definitions or axioms. As you might have heard, there are many debates on foundations of mathematics in Europe these days.

Zhu : Principles (*li* 理) do not serve the purpose of justifying truth, they are rather immanent to mathematical objects, whose meaning can be elucidated by recognizing their structural properties and patterns.

Russell : That is futuristic, Hilary Putnam will love this, mathematics without foundations !

Zhu : How dare you say that, you moron ! Our mathematics is well grounded, august Emperors from Zhou dynasty have laid out their basis. I feel truly insulted by you, Sir. Goodbye.

The fictive interview between two historical actors who have never met, Bertrand Russell (1872-1970), a British mathematician, logician and philosopher, and Zhu Shijie 朱世傑 (active around 1300 CE), a Chinese Yuan 元 dynasty (1271-1368) author of two mathematical books, illustrates the absurdities of searching for an equivalent of what only in the 20th century became the field of philosophy of mathe-

3. It is often assumed, that in the West there was a philosophy of mathematics as a field of inquiry early on. That this was not the case, but that there were nevertheless philosophical aspects to be found in Greek mathematics is discussed in Fabio ACERBI, "Two Approaches to Foundations in Greek Mathematics : Apollonius and Geminus", *Science in Context*, 23, 2 (2010), p. 151-186.

4. Cf. the former preface to ZHU Shijie's *Jade Mirror of Four Unknowns* (*Si Yuan Yujian Qian Xu* 四元玉鑒前序) : *Shu yi eryi. Yi zhe, wanwu zhu suo cong shi* 數一而已. 一者, 萬物之所從始.

matics. The above dialogue's aim is not to juxtapose a Western philosophy of mathematics to an inexistent counterpart in China, neither is its purpose to ridicule attempts of contemporary historians to describe mathematical practices in China from a philosophical point of view based on a few cherry-picked traces : they would all agree that there ARE philosophical aspects to be found in mathematical texts and even more so in their commentaries.[5] When I say "philosophical", I mean traces of both kinds of reflections : the object of mathematical inquiry and the mathematician's toolbox, which are his methods, practices of argumentation and classification and linguistic tools and standards developed to speak about mathematical concerns. All of these aspects we do find built into the corpus of pre-modern Chinese mathematics,[6] yet, nothing explicit is found outside of it in the form of a meta-discourse on mathematics itself. A rare exception is Hua Hengfang's 華蘅芳 (1833-1902) *Brush Talks on Mathematical Learning* (*Xue Suan Bitan* 學算筆談), a collection of notes and short essays on mathematical concerns. In one short essay on the principles of mathematical methods (*suanfa zhi li* 算法之理), Hua assumes that at a certain stage of development, we have primary cognitive faculties for supporting our quantitative senses, which, when fine-tuned by learning mechanisms, allow for more efficient processing of numerical problems. Hua juxtaposes "methods" (*fa* 法) with "principles" (*li* 理), the latter being a well-known term from philosophical discourses and of particular interest in this article with respect to mathematics.[7] As I will show for a certain mathematical domain, "mathematical principles" there are the abstract foundations of mathematical objects and tools, including argumentative patterns among the latter. Contrary to some authors who interpret the term as referring to a specific kind of argumentative style in 17th and 18th century Chinese mathematics,[8] I do not

5. Karine Chemla has written extensively about the topic for ancient China and its foundational canon, the *Nine Chapters on Mathematical Procedures* (*Jiu Zhang Suan Shu* 九章算術) from Han dynasty and the commentary by Liu Hui from 263. Cf. her contributions in Karine CHEMLA, GUO Shuchun, *Les Neuf chapitres sur les procédures mathématiques*, Paris, Dunod, 2004, p. 1070-1073.

6. The notion of a "mathematical" field or corpus used here is limited to what in the *Draft History of the Qing* (*Qingshi Gao* 清史稿) were "writings on calculations belonging to the category of astronomical and mathematical methods" (*tianwen suanfa lei suan shu zhi shu* 天文算法類算書之屬). I do leave aside in this article all writings that were listed as "numerical learning belonging to the category of procedures with numbers" (*shushu lei shuxue zhi shu* 術數類數學之屬), i.e. the numerological tradition. This does not mean that these categories were precisely delineated. For example, the category Images and Numbers (*Xiangshu* 象數) in the *Encyclopedia of Numerical Learning Old and New* (*Gujin Suanxue Congshu* 古今算学叢書, 1898), includes, among many other books that we would consider "mathematical" even from a modern point of view, Li Shanlan's 李善蘭 *Duoji Bilei* 垛積比類 (Comparable Categories of Discrete Accumulations), which I rely on heavily in this article. Cf. LI Shanlan, *Duoji Bilei*, in *Zeguxizhai Suanxue* 則古昔齋算學 (Mathematics from the Zeguxi Studio), vol. 4, Haining, 1867.

7. Within the framework of this special issue, it is not the goal of my article to analyze historically the conceptual changes in the appropriation of *li* from a philosophical to a mathematical context, a topic which deserves a separate study. Although *li* figures prominently in the title, HO Peng-Yoke, *Li, Qi and Shu. An Introduction to Science and Civilisation in China*, Hong Kong, Hong Kong University Press, 1985, does not discuss the term in mathematics more specifically. A good overview of its significance in Neo-Confucian thought is still Joseph NEEDHAM, *Science and Civilization in China*, vol. 2, Cambridge, Cambridge University Press, 1956, p. 472-485, for earlier, Daoist conceptualizations rooted in the observation of natural jade fractures along its patterned veins, see Harold D. ROTH, "The Classical Daoist Concept of LI (Pattern) and Early Chinese Cosmology", *Early China*, 35 (2013), p. 157-183.

8. "Never the kind of pure deduction in the Euclidean manner, but *a combination of induction and deduction*, with the help of intuition, for the purpose of problem solving." See SU Jim-Hong and YING Jia-Ming,

believe that there is such unique interpretation given to the term by the actors themselves, but a variety of meanings dependent on the specific contexts to which *li* is applied.[9] Hua, for example, relates his discussion to mathematical cognition by assuming that children know the "principles" before they can measure and count :

> If in a person's mind there is indeed ignorance and no awareness, then there is no need to discuss the learning of mathematics. But if there is a little awareness, and the person is able to think and argue, then he already possesses the principles (*li* 理) for mathematical learning. It is innate, just try to observe children at play : when they see the fruit, they will necessarily strive for the biggest one, because their brain already has the capacity to perceive magnitudes. From this we know that the principles of mathematical learning (*suanxue zhi li* 算學之理) are inherent to our minds (*ren xin suo zi you* 人心所自有) and do not come from outside (*zi wai* 自外). Therefore, if we select some not very arduous problems from a mathematical book, those who were not trained in mathematics with words, can also, by bringing together their thoughts, find the number that is asked for. Yet, when the degree of difficulty is gradually picking up, then it is much easier for those who master mathematics. That is because if for the calculation one does not yet have a method, then all the numbers need to be entirely reckoned by applying one's mind. Necessarily, this is truly difficult ! But if one knows the mathematical method (*suanfa* 算法), then no matter how the numbers are set up, everything can be solved by applying the method and it is not necessary to apply one's mind to it in order to reduce the work and obtain twice the effect. I believe that any mathematical method at the outset entirely emerges according to mathematical principles (*suanli* 算理). It is only once that the method is obtained that principles do reside within the method (*li ji yu yu fa zhi zhong* 理即寓于法之中). One can, by engaging with a method, obtain the principles, and one can also by setting aside the principles apply a method. If the method does not fail, then the principles cannot be erroneous either ![10]

To clarify what "principles" concretely refer to in one specific field of mathematical inquiry, I will focus in the following on a branch of mathematics that evolved into a true discipline in the 19th century : "discrete accumulations" (*duoji* 垛積). Involving natural numbers only, my discussion will elaborate upon all aspects raised in the introductory dialogue. I will first explain what "discrete accumulations" are and show how through structural adjustments of text (i.e. by changing the order of certain problems) they turned into an object of mathematical inquiry. In a second step, I will analyze reflections upon a mathematician's toolbox for dealing with these "discrete accumulations". The main historiographical argument of this paper is to show that philosophical aspects of mathematics in China are sparse in their manifestation and

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"What Did They Mean by 'Calculation Principles' ? : Revisiting Argumentative Styles in Late Ming to Mid-Ching Chinese Mathematics", *Korean Journal for the History of Science*, 38, 2 (2016), p. 351-376, here p. 375.

9. In Jiang-Ping Jeff CHEN, "Practices of Reasoning : Persuasion and Refutation in a Seventeenth-Century Chinese Mathematical Treatise of 'Linear Algebra'", *Science in Context*, 33 (2020), p. 65-93, for example, it is shown that for the Qing dynasty mathematician Mei Wending 梅文鼎 (1633-1721) the epistemological values of uniformity, simplicity, and rigor constitute "the core essence of *suanli* 算理 (principles of computations)" (p. 66).

10. Translated according to the essay "General Discussion of the Principles of Mathematical Methods" (*Zonglun Suanfa zhi Li* 總論算法之理), in HUA Hengfang, *Xue Suan Bitan* 學算筆談 (Brush Talks on the Study of Mathematics), 2 vol., in *Xingsu Xuan Suangao* 行素軒算稿 5, Liangxi Hua Shi Zangban 梁谿華氏藏版 ed., 1885, p. 1.1a-1.1b (punctuation is mine).

dispersed between texts, paratexts and images, borrowing concepts from other Chinese contexts of philosophical nature. My argument builds upon two such concepts which play an important role in the field of "discrete accumulations" : "principles" (*li* 理) and "comparable categories" (*bilei* 比類).

General "principles" underlying mathematical objects and procedures there refer to common structural patterns in diagrams and procedures related to sequences of numbers. Recognizing these patterns, as some authors claim, allows to conjecture general procedures by analogy and (incomplete) induction. Although not explicitly spelled out as a valid mode of argumentation, by the 19th century, standard linguistic formulations and diagrammatic codes attest of an established set of discursive and visual elements in mathematical writings for expressing common patterns in mathematical objects and procedures of "comparable categories". Such scholarly tools set forth to deal with "discrete accumulations", as I will show, can be considered the reflection of philosophical considerations on the very nature of the underlying mathematical "principles" common to an entire set of mathematical objects and procedures.

II. "DISCRETE ACCUMULATIONS" AS MATHEMATICAL OBJECTS

Li Shanlan 李善蘭 (1811-1882), the author of a 19th century treatise entitled *Comparable Categories of Discrete Accumulations* (*Duoji Bilei* 垜積比類, 1867) sees his own work on the kind of mathematical object designated by "discrete accumulations" as the establishment of a true field of mathematical knowledge outside the canonical tradition :

> I want those who learn mathematics to know that the procedures for discrete accumulations erect another flag beyond the *Nine Chapters*. Their theory has begun with no other than myself (*yu ling xi suanjia zhi duoji zhi shu yu Jiuzhang wai bie li yi zhi, qi shuo zi Shanlan shi* 欲令習算家知垜積之術於九章外別立一幟，其說自善蘭始) ![11]

Li's book relates to summations of finite series, but research on this mathematical subject was far from being a novelty in his time. Since the canonical *Nine Chapters*, it has been approached in different contexts, within geometry first under the Han 漢 (202 BC-220 AD), but also in astronomical and algebraic contexts under the Yuan. The very expression "discrete accumulations", or literally "accumulated heaps", gives a name to a real field of mathematical research in pre-modern China. The choice to translate the expression by "discrete accumulations" reflects the strong links that this field has with geometry, and in particular with figurate numbers. The idea is to accumulate unitary — and therefore countable — elements by forming certain geometric objects whose volume (*ji* 積) is known for the case of a continuous space.

11. Translated from LI Shanlan, *Duoji Bilei*, 1.1a. For a complete translation to French, cf. LI Shanlan, *Les Catégories analogues d'accumulations discrètes*, introduction, critical edition and translation by Andrea BRÉARD, Paris, Les Belles Lettres (forthcoming 2023).

1. Shared patterns underlying structural principles

The first seeds of this type of rapprochement of the arithmetically discrete and the geometrically continuous can be found in one chapter of the *Nine Chapters*. A series of twenty-two problems asking to calculate the volume of certain geometric shapes is followed by three problems where grains are piled up in a corner or against a wall. One asks for the volume of the heap — not for the number of grains piled up. In the solution the same procedure as for a circular cone given earlier in the chapter is applied. This fact is mentioned in the commentary by Liu Hui 劉徽 (263). What seems like a simple side remark turns into a central conceptual stance in light of the later history. Looking for example at Song 宋 dynasty (960-1279) commentaries one is rather inclined to read Liu Hui's statement as a testimony of classificatory reflections upon mathematical objects : although a heap of grains and a continuous volume is constituted differently and thus has a different inner structure, yet both mathematical objects share the same procedure for calculating their volume.

Here is the later history of "discrete accumulations" in more detail to confirm my point : It is actually under the Song that a more systematic elaboration by Yang Hui 楊輝 (ca. 1238-1298)[12] is attested in the transmitted sources. Like Li Shanlan later on, Yang proceeds by "comparable categories" (*bilei* 比類),[13] for example, when he juxtaposes the calculation of the volume of a pyramid with a square base with the summation of the square numbers. In Yang Hui's work, objects are classified and explained by being compared or assimilated to each other, and the related problems are then solved by similar or even identical procedures. His book *Detailed Explanations of the Nine Chapters on Mathematical Methods* (*Xiangjie Jiuzhang Suanfa* 詳解九章算法), printed in 1261, therefore contains mathematical problems from the *Nine Chapters* which are rearranged and supplemented by problems of "comparable categories". The following example illustrates how the procedural similarity between the new analogous problem and its counterpart in the canonical book is established.

While the *Nine Chapters* provided the procedure for a problem asking to calculate the volume of a truncated pyramid with a rectangular base, called *chutong* 芻童 (lit. a haystack), Yang Hui added a problem of "comparable category" in which the same geometrical shape (except that it is mirrored horizontally) is not a solid volume but constructed from discrete elements. The question of the problem is thus no longer what would be the volume of the resulting solid, but rather what is the total number of unitary objects that are stacked in the form of a truncated pyramid with a rectangular base.[14] Yang Hui even explains in the solution procedure, which sequences of operations remain unchanged, and which operations modify the original method[15] :

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12. Cf. Andrea BRÉARD, *Re-Kreation eines mathematischen Konzeptes im chinesischen Diskurs : Reihen vom 1. bis zum 19. Jahrhundert*, Stuttgart, Steiner, 1999, chap. 1 and 3.
13. I have chosen to translate the expression *bilei* 比類 literally as "comparable categories" and not as usually done by "analogy", in order to avoid proximity with the Greek conception of analogy as equality of ratios.
14. We find the same approach, applied to the determination of the area of the regular polygons, from the equilateral triangle to the dodecagon, in the *Extracts by Epaphroditus and Vitruvius Rufus*, a Roman

> Suppose we have a *chutong*. Its lower width [*a*] is 2 *zhang*, its length [*b*] is 3 *zhang*, its upper width [*c*] is 3 *zhang*, its length [*d*] is 4 *zhang*, its height [*h*] is 3 *zhang*. The question is : what is its volume (*ji* 積) ?
> The answer says : 26500 *chi.*[16]
> Explanation of the problem : [the shape] is similar to an observation platform, lengthened in length.
> The calculation says : Doubling the upper length of 4 *zhang* makes 80 *chi*. Adding the lower length of 30 *chi* makes 110 *chi*. Multiplying this by the upper width of 30 *chi*, we get 3300 *chi*. Doubling the lower length of 3 *zhang* makes 60 *chi*. Adding the upper length of 40 *chi* makes 100 *chi*. By multiplying this by the lower width of 20 *chi*, we get 2000 *chi*. By adding the two positions together, we get 5300 *chi*. Multiplying this by the height of 30 *chi* gives 159000 *chi*. Dividing this by 6 gives 26500 *chi*. This corresponds to what has been asked for.
> The comparable category (*bilei* 比類) : a pile of seeds. An upper length of 4, a width of 2, a lower length of 8, a width of 6, a height of 5. The question is : how many in total ? The answer is : 130. The method says : Double the upper length, add the lower length. Multiply this by the upper width and you get 32. Also, double the lower length, add the upper length. Multiplying the lower width by this value gives 120. When the two positions are added together, the result is 152. This is the original method that governs the volume of a *chutong*. With the upper length, we reduce the lower length, we also add the remaining 4. A stack of seeds is therefore not equal to a circular object, nor to a rectilinear volume. This is why this term must be added. By the height, we multiply this and [we get] 780. The division by 6 is also part of the original method of a *chutong*.

The procedures indicated in the text above correspond to the following calculations for the volume of the *chutong* in the continuous case :

$$\mathrm{V} = [(2\mathrm{d} + \mathrm{b}) \cdot \mathrm{c} + (2\mathrm{b} + \mathrm{d}) \cdot \mathrm{a}] \cdot \mathrm{h} \div 6$$

and to :

$$\mathrm{A} = [(2\mathrm{d} + \mathrm{b}) \cdot \mathrm{c} + (2\mathrm{b} + \mathrm{d}) \cdot \mathrm{a} + (b - d)] \cdot \mathrm{h} \div 6$$

for the accumulation in the "comparable" discrete case. Due to the parallel formulation of the operations to be carried out in both cases, the corrective term (b − d) for the discrete case with respect to the continuous can easily be recognized.[17] This type of textual organization, where old mathematical problems are confronted with new mathematical objects and rearranged according to procedural similarities and structural analogies, is more systematically applied by Yang Hui in his book, *Rapid Methods of Multiplication and Division Compared to Categories of Fields and [their]*

agrimensoric text. Cf. Nicolas BUBNOV, ed., *Gerberti Opera Mathematica*, Berlin, R. Friedländer & Sohn, 1899, p. 534-545.

15. Translated from YANG Hui 楊輝, *Xiangjie Jiu Zhang Suanfa* 詳解九章算法 (Detailed Explanations of the Nine Chapters on Mathematical Methods), Yijiatang congshu 宜稼堂叢書 ed., 1842, 78a-78b.

16. Measure of length, whereby 1 *zhang* = 10 *chi*. Linguistically, no difference is made in the Chinese text between measures of length, surface or volume, i.e. between *chi*, square *chi* or cube *chi.*

17. In a geometric interpretation, this corrective term is dimensionally incorrect. A reconstruction of its origin shows that one must think of the "surface" (b − d) as multiplied by a third dimension equal to unity. Cf. Andrea BRÉARD, "A Summation Algorithm from 11th Century China. Possible Relations Between Structure and Argument", in Arnold BECKMANN, Costas DIMITRACOPOULOS, Benedikt LÖWE, ed., *Fourth Conference on Computability in Europe*, New York, Springer, 2008, p. 77-83 for the justification of the algorithm for a *chutong* in the discrete case discussed by Shen Gua 沈括 (1031-1095) about a century before Yang Hui.

Measurements (*Tianmu Bilei Chengchu Jiefa* 田畝比類乘除捷法) of 1275.[18] When Zhu Shijie included in 1303 in his *Jade Mirror of the Four Unknowns* (*Siyuan Yujian* 四元玉鑑) four chapters entirely devoted to the problems of "discrete accumulations", the mathematical domain under the same name took on well-defined contours.[19] Even more so under the Qing 清 (1644-1912), when an important number of commentaries were written following the rediscovery of Zhu Shijie's treatise. The field of "discrete accumulations" became one of the major themes of mathematical research, partly influenced by Western mathematics from the end of the 19th century on.[20]

Title		Author	Year
Shaoguang Buyi 少廣补遺	*Supplement to* [the Chapter] *"Decreasing the Width"*[21]	Chen Shiren 陳世仁 (1676-1722)	ca. 1720
Dijian Shuli 遞兼數理	*Mathematical Principles of Sequential Combinations*	Wang Lai 汪萊 (1768-1813)	ca. 1796-1805
Duiduo Qiuji Shu 堆垛求積術	*Procedures for Finding the Accumulation of Heaps of Piles*	Dong Youcheng 董祐誠 (1791-1823)	Preface 1821
Duiduo Ceyuan 堆垛測圓	*Heaps of Piles and Circle Measurement*	Xu Youren 徐有壬	ca. 1840-1859
Chaocha Shujie 超差術解	*Explanation of the Procedure of Divided Differences*	Fu Jiuyuan 傅九淵 (Jinshi 1823)	Printed in 1888/1889
Duoji Bilei 垛積比類	*Comparable Categories of Discrete Accumulations*	Li Shanlan 李善蘭	1867

18. In the same work, another kind of diagrams with similar graphic features, i.e. grids composed of unit squares, are used by Yang Hui to depict the transformations during the calculation of surfaces, cf. Alexei VOLKOV, "Geometrical Diagrams in Traditional Chinese Mathematics", in Francesca BRAY, Vera DOROFEEVA-LICHTMANN, Georges MÉTAILIÉ, ed., *Graphics and Text in the Production of Technical Knowledge in China. The Warp and the Weft*, Leiden, Brill, 2007, p. 445-448.

19. In addition, we can see the establishment of a close link of the domain with the arithmetic triangle, which was placed at the very beginning of Zhu Shijie's book, and which also opens the ball for the sequence of triangles in Li Shanlan's *Comparable Categories of Discrete Accumulations*. This triangle, in the West known as the Pascal Triangle, can be found for the first time in China in Yang Hui's *Detailed Explanations* (1261), but we know that it must have circulated approximately a century earlier in Jia Xian's 賈憲 (ca. 1010-1070) work, not to speak of its earlier occurrences in Byzantine, Sanskrit, and Arabic sources.

20. On this influence, cf. TIAN Miao, "The Westernization of Chinese Mathematics. A Case Study of the *Duoji* Method and its Development", *East Asian Science, Technology, and Medicine*, 20 (2003), p. 45-72.

21. It is particularly interesting to note that the first work indicated, that of Chen Shiren, refers to the title of one chapter of the *Nine Chapters*. It deals with the extraction of the square root in a geometric context and Li Shanlan indicates precisely this same chapter as the origin of studies on "discrete accumulations" without any filiation being explicated. Yet, both aim at determining a quantity of a plane or solid geometric object : in the *Nine Chapters* this object is characterized by continuous quantities, while in the category of "discrete accumulations" these are objects formed by discrete points. Certainly, around 1720 Chen Shiren probably had no choice but to take the *Nine Chapters* as the starting point for an original contribution, other texts being still lost in his time, in particular those of Zhu Shijie. See *Zhejiang Chouren Zhushu Ji* 浙江疇人著述記 (Bibliographic Notices on Mathematicians and Astronomers from Zhejiang Province), cited by DING Fubao 丁福保 and ZHOU Yunqing 周雲青, ed., *Sibu Zonglu Suanfa Bian Shumu* 四部總錄算法編書目 (Complete Chronicle in Four Sections, Bibliography of Mathematical Editions), Shanghai, Shangwu yinshu guan, 1957, p. 70b : "The Supplement to the [chapter] 'Decreasing the Width' sheds particular light on methods of calculation for discrete accumulations. In its time, the mathematical books of the Song and Yuan had fallen into oblivion and were no longer in circulation. As a result, there were no mathematicians or astronomers who studied and mastered the procedures for discrete accumulations in his time. Shiren was endowed with an extraordinary character, he himself invented many things in the field of this science !"

Duoji Yanjiao 垛積演較	*Discrete Accumulations Deduced by Comparison*	Hua Hengfang 華蘅芳	1896
Duoji Yi De 垛積一得	*Discrete Accumulations Obtained Easily*	Cui Chaoqing 崔朝慶	Manuscript printed in 1898 in the encyclopedia 古今算學叢書

Table 1. Chronological list of books placed in the category "Discrete Accumulations" of the *Combined Commentary on the Bibliography of Mathematical Books, Ancient and Recent and the Preliminary Compilation of Mathematical Studies* (*Gujin Suanxue Shulu Suanxue Kao Chubian Hezhu* 古今算學書錄算學考初編合注), 1957.[22]

The above list gives some titles of works which, by the actors themselves or in the *Biographies of Astronomers and Mathematicians*, are declared to be part of the field of research on "discrete accumulations" (see table 1). But besides having in common from a pragmatic point of view the same object of study, what was it that provided unity to this field ? In absence of an explicit answer provided by the compilers of bibliographies, it is best to turn again to the mathematical practitioners.

2. Interconceptuality between mathematics and philosophical traditions

Li Shanlan himself is said to have made a distinction between numbers (*shu* 數) and principles (*li* 理), claiming that "although numbers have a myriad transformations, their principle alone is the fundamental procedure !" (*shu you wan bian li wei yuan shu* 數有萬變理惟元術).[23] Confronting this paradigm with Li's work on *Discrete Accumulations*, it seems indeed to be underlying his conception of finding a unique, general algorithm for the different kinds of number sequences in the diagonals of the kind of arithmetic triangles which he presents : based on the first four or five summation procedures, Li Shanlan systematically asks the reader to induce the general method by analogy on the basis of "comparable categories" (*bilei* 比類),[24] thus assuming that there is a single pattern underlying all of them. As mentioned above, the expression *bilei* has already been at the core of Yang Hui's way of organizing continuous and discrete geometric objects into procedurally connected pairs.[25]

22. *Ibid*., p. 70b-71a. Far from being exhaustive, this list contains few works published after Li Shanlan's 1867 book on "Discrete Accumulations".

23. Cf. ZHU Kebao, *Chouren Zhuan San Bian* 疇人傳三編 (Biographies of Astronomers and Mathematicians 3), in WANG Xianqian, ed., *Nanjing Shuyuan Congshu*, 3-5 (1888), p. 6.25a.

24. Li Shanlan actually refers to induction by "analogical extension" (*leitui* 類推), yet another philosophical term, particularly prominent in Neo-Confucian philosophy in Song dynasty. Cf. KIM Yung Sik, "'Analogical Extension' (*leitui*) in Zhu Xi's Methodology of 'Investigation of Things' (*gewu*) and 'Extension of Knowledge' (*zhizhi*)", *Journal of Song-Yuan Studies*, 34 (2004), p. 41-57.

25. The philosophical term can be found in early Chinese texts such as *The Classic of Rites* (*Liji* 禮記, 475-221 BCE), where one finds the following reference to quantitative similarity between different classes : "In this month, orders are given to the officers of slaughter and prayer to go round among the victims for sacrifice, seeing that they are entire and complete, examining their fodder and grain, inspecting their condition as fat or thin, and judging of their looks. They must arrange them according to their classes. In measuring their size, and looking at the length (of their horns), they must have them according to the (assigned) measures. When all these points are as they ought to be, God will accept the sacrifices". Cf. the chap. "Proceedings of Government in the Different Months" (*Yue Ling* 月令). Translation from James LEGGE, *Book*

Li Shanlan goes a step further by looking at more than two objects that share a common algorithmic pattern and inductively seeks a general procedure for an entire sequence of objects.

That classes of mathematical objects do share a common principle, is not a philosophical *topos* restricted to "discrete accumulations", neither is the habit of relying on terms and concepts borrowed from other fields of intellectual inquiry. Combinatorics, indeterminate analysis, or, as the example of conic sections below will show, were all domains that fell for example back on notions from the *Changes.*[26] When Xia Luanxiang 夏鸞翔 (1823-1864), a contemporary of Li Shanlan, gives a summary of his 1861 work on conic sections, he introduces the subject matter of his book as follows :

> Heaven is big and round. When describing the objects of heaven, there is none that does not relate to the circle. Although "circle" is only a unique name, there are a myriad of species. When following the circle for one round, curves are generated. Westerners divide [the different kinds of] lines according to the order of generation into the following categories : lines of the first order, [this class consists of] the straight line only. Lines of the second order, [this class comprehends] four species : the circle, the ellipse, the parabola and the hyperbola. Lines of the third order have eighty different kinds. Lines of the fourth order have more than five thousand kinds. Beyond the lines of the fifth order, it is such that it is impossible to investigate them.[27] Here, I explore the four kinds of lines of the second order and trace their origins, and additionally, I add explanations to all the parabolas of higher order.[28] *Although their forms have a myriad variations, their principles are from a single strain* (*xing sui wan shu li shi yi guan* 形雖萬殊理實一貫). All the equations of conics are entirely provided for on the solid [i.e. the surface] of a cone. That is the reason why the cone is the mother of the curves of the second order.[29] For the ellipse one uses congregation, for the parabola one uses extension, for the hyperbola one uses dispersion, but their principles all stem from the plane circle. If indeed, we unite what they have in common, then we "build tools to imitate the cosmos" (*zhiqi shangxiang* 制器尚象).[30] By "bowing [to the earth] and looking upwards [to the heaven], by observing [the sky]

of Rites, 2 vol., Oxford, Clarendon Press, 1885, Section III, Part II.9. Another example can be found in the chap. "Record of Music" (*Yue ji* 樂記) : cf. *ibid*., II.14.

26. For combinatorics, cf. Andrea BRÉARD, "Hexagrams and Mathematics. Symbolic Approaches to Prediction from the Song to the Qing", in Tze-ki HON, ed., *The Other Yijing. The* Book of Changes *in Chinese History, Politics, and Everyday Life*, Leiden, Brill, 2021, p. 192-220. For indeterminate analysis, cf. Ulrich LIBBRECHT, *Chinese Mathematics in the Thirteenth Century*, Cambridge, MIT Press, 1973, chapter 15, p. 267-293.

27. This passage is reminiscent of Isaac Newton's classification of curves that is related in more detail by Elias Loomis (1811-1889) in his *Elements of Analytical Geometry, and of Differential and Integral Calculus* (Elias LOOMIS, *Elements of Analytical Geometry and of Differential and Integral Calculus*, New York, Harper & Brothers, 1851) and translated into Chinese by Li Shanlan and Alexander Wylie (1815-1887). Cf. LI Shanlan, Alexander WYLIE 偉烈亞力, trans., *Dai Weiji Shiji* 代微積拾級, Shanghai, Mohai shuguan, 1859, vol. 8, p. 6b-7b.

28. I.e. the cubical parabola, the biquadratic parabola, etc.

29. Cf. E. LOOMIS, *Elements of Analytical Geometry and of Differential and Integral Calculus*, p. 103 : "[…] the only curves whose equations are of the second degree, are the circle, parabola, ellipse, and hyperbola".

30. This expression is borrowed from The Great Appendix (*Xici* 繫辭) to the *Classic of Changes*. For a translation, cf. James LEGGE, *The I Ching*, New York, Dover, 1963, app. III, chap. X.59, p. 367-369.

and investigating [the ground]" (*fu yang guang cha* 俯仰觀察),[31] their applications are without limitations (*weiyong wuqiong yi* 為用無窮矣).[32]

It is perhaps surprising that a Chinese mid-nineteenth-century mathematician who was familiar with translations of English works on analytic geometry, differential and integral calculus, as well as with the works of his Chinese predecessors, frames philosophically a complex mathematical topic that he deals with using a technically original and novel approach. That Xia wanted to promote his work among Confucian scholars with a preface that proved that he was versed in the classics and familiar with the *Classic of Changes*, is not unusual in paratextual discourse. But Xia certainly also wanted to understand mathematics as a numerical science that studies change. Analogous to the transformations of broken into unbroken lines in the divinatory hexagrams, conics, as he underlines in the above-quoted passage, can be obtained from a single origin, the cone, and transformed one into another : the ellipse can be obtained by joining together the two extremities of the parabola, and if one extends one endpoint of the major axis of the ellipse to infinity, it is transformed into a hyperbola.[33] Yet, this was clearly a philosophical thought experiment, since strictly speaking mathematically, in order to perform a mapping from an ellipse to a hyperbola or parabola a projective transformation is needed. No affine transformation can change a bounded curve into an unbounded one.[34]

Xia's statement that "although their forms have a myriad variations, their principles are from a single strain" (*xing sui wan shu li shi yi guan* 形雖萬殊理實一貫) closely relates to the similar phrase by Li Shanlan quoted at the beginning of this section, "although numbers have a myriad transformations, their principle alone is the fundamental procedure" (*shu you wan bian, li wei yuan shu* 數有萬變理惟元術), yet on a more visual note. The myriad manifestations of a single entity are a common trope in both, Confucian and Daoist philosophy,[35] it is thus difficult to link these

31. The common phrase *yang guan fu cha* 仰觀俯察 has its origins in The Great Appendix to the *Classic of Changes*. For a translation, cf. *ibid*., app. III, chap. IV.21, p. 353 : "[The sage], in accordance with [the *Changes*], looking up, contemplates the brilliant phenomena of the heavens, and, looking down, examines the definite arrangements of the earth" (*yang yi guan yu tianwen, fu yi cha yu dili* 仰以觀於天文，俯以察於地理).

32. Translated from XIA Luanxiang, *Zhiqu Tujie* 致曲圖解 (Diagrammatic Explanations [of Procedures] for Curves), in Xuxiu Siku Quanshu Bianwei Hui, ed., *Xuxiu Siku Quanshu* 續修四庫全書 (Supplement to the Complete Books in the Four Treasuries), vol. 1047, Shanghai, Shanghai guji chubanshe, 2006, p. 438. The italics are mine.

33. On Xia's ideas concerning a comprehensive treatment of conics, cf. LIU Dun, "Xia Luanxiang Dui Yuanzhui Quxian de Zonghe Yanjiu 夏鸾翔对圆锥曲线的综合研究" (The Comprehensive Research on Conic Sections by Xia Luanxiang), in DU Shiran, ed., *Di San Jie Guoji Zhongguo Kexueshi Taolunhui Lunwenji* 第三届国际中国科学史讨论会论文集 (Proceedings of the Third International Conference on the History of Chinese Science), Beijing, Kexue chubanshe, 1990, p. 13.

34. Projective transformations map lines to lines but do not necessarily preserve parallelism, whereas affine transformations respect parallel lines. The latter allow, for example, mapping a circle to an ellipse, but not an ellipse to a parabola or hyperbola.

35. See for example Guo Xiang's 郭象 (252-312) commentary on the Zhuangzi in 《莊子翼·齊物論》 : "although their [things'] forms have a myriad variations, they are one in attaining their own determinacies, thus [the text] says, 'the Way makes them all into one' " (形雖萬殊而性同得故日：道通為一也), or Xunzi 《荀子·儒效》 : "through a thousand affairs and ten thousand changes, his [a Great Confucian's]

statements to any specific school of thought. The kind of intertextuality observed for both authors nevertheless confirms that mathematics was not an isolated knowledge system but participated in processes of conceptual transfers, and even possibly influenced the formation of technical terms in the late Qing which translated foreign concepts. Induction, as a 1913 *Dictionary of Philosophical Terms* shows, was then translated as *wan shu yi ben* 萬殊一本, literally meaning "a myriad transformations, a single origin".[36]

III. A MATHEMATICIAN'S TOOLBOX FOR "DISCRETE ACCUMULATIONS"

To understand better the philosophical implications of "principles" (*li* 理) specifically in mathematical writings, one needs to go beyond narratives and include their visual elements, too. In the above impossible interview with Russell, Zhu Shijie's statement about the visibility of "principles" in diagrams (*tu* 圖), and thus about a cognitive foundation for understanding mathematical knowledge, was not entirely fictional. In the text that accompanies a diagram showing the geometric configuration obtained from squaring the magnitudes associated to four unknowns at the beginning of Zhu's *Jade Mirror* (1303), he concludes by saying that "if one studies the diagram, one will recognize this.[37] Its principle is of evident nature" (*kao tu ren zhi qi li xian ran* 考圖認之, 其理顯然).[38] Turning more specifically to figured numbers in China and their representations, one can thus ask what visualized "principles" signify in a number theoretical context.

Visual representations of numbers are found abundantly since the Song period in philosophical writings, most prominently in relation to *The Chart of the Yellow River* (*Hetu* 河圖) and *The Writ of the Luo River* (*Luoshu* 洛書) diagrams,[39] in which numbers are shown as composed of black or white pebbles, each corresponding to a unit.[40] Beginning with Yang Hui, figured numbers, so-called "heaps" (*duo* 垛), ap-

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Way is one" (千舉萬變, 其道一也). The latter translation is from John KNOBLOCK, *Xunzi*, vol. II, Stanford, Stanford University Press, p. 79.

36. Timothy RICHARD, Donald MACGILLIVRAY, ed., *Dictionary of Philosophical Terms. Chiefly from the Japanese*, Shanghai, Christian Literature Society for China, 1913, referred to in Joachim KURTZ, *The Discovery of Chinese Logic*, Leiden, Brill, 2011, p. 415.

37. I.e. that one obtains $x^2 + y^2 + z^2 + w^2 + 2xy + 2yz + 2zw + 2wx + 2xz + 2yw$ when squaring $x + y + z + w$.

38. Cf. the *Diagram for the Calculation of Segments When Squaring the Four Unknowns* (*Si Yuan Zi Cheng Yan Duan Zhi Tu* 四元自乘演段之圖) in the introductory section of Zhu Shijie's *Jade Mirror of Four Unknowns* (*Si Yuan Yujian* 四元玉鑒).

39. The earliest extant illustrations of these diagrams date from the tenth century, see Richard J. SMITH, *Fathoming the Cosmos and Ordering the World. The Yijing (I- Ching or Classic of Changes) and Its Evolution in China*, Charlottesville, University of Virginia Press, 2008, p. 79-80 and 117-119.

40. For philosophical contexts in which pebble diagrams appear during the Song, see for example Holger SCHNEIDER, *Aspekte diagrammatischer Argumentation im Werk des Liu Mu der chinesischen Songzeit*, PhD thesis, Friedrich-Alexander-Universität Erlangen-Nürnberg, 2017 ; and the commentary to Zhu Xi's 朱熹 (1130-1200) *Primer of Yi Study* (*Yixue Qimeng* 易學啟蒙, 1186) by LI Guangdi 李光地 (1642-1718), *Qimeng fulun* 啓蒙附論, in LI Guangdi, ed., [*Yuzuan*] *Zhouyi Zhezhong* 〔御纂〕周易折中, *juan* 21, SKQS, 經部 vol. 38, p. 521-553.

pear in diagrams as "comparable categories" to continuous plane geometric objects, so-called "fields" (*tian* 田) in the shape of squares, circles, triangles and trapezoids.[41] Next to the diagrams in figure 1 we are told that "for all [the surfaces shown] the method for the trapezoidal field can be applied. One does not need to establish yet another problem with a calculation sketch". In the absence of further explanations, one might ask what justifies, for example, that a topsy-turfy trapezoid or the number of pebbles in the triangle composed of lines with 1 to 7 pebbles are calculated by the same procedure as the surface of the trapezoid.[42] That Yang Hui has grouped together all four diagrams seems indeed to imply that he intends to show a structure and properties common to all of them.

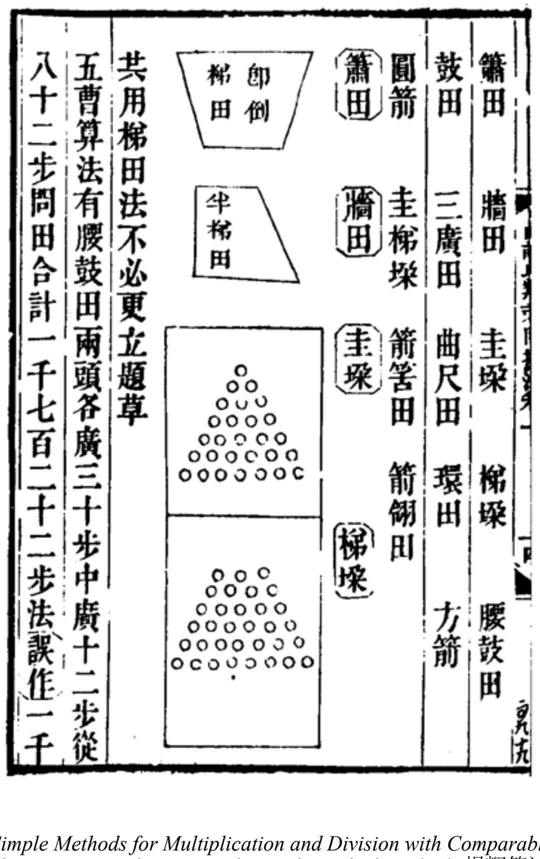


Figure 1 : *Simple Methods for Multiplication and Division with Comparable Categories in Field Measurement*, in Yang Hui's Mathematical Methods 楊輝算法 (1275).

As for three-dimensional figured numbers, algorithms for calculating the number of wine jars piled up in the shape of a truncated pyramid with a rectangular base were

41. Cf. Andrea BRÉARD, "What Diagrams Argue in Late Imperial Chinese Combinatorial Texts", *Early Science and Medicine*, 20, 3 (2015), p. 241-264.

42. Here $(7 + 1) \cdot 7 \div 2$.

first stated by Shen Gua in the 11th century.[43] Although such algorithms circulated widely in Ming 明 (1368-1644) dynasty mathematical manuals, even in versified form for easy memorization,[44] I have not found a single illustration of such three-dimensional forms before the *Essence of Numbers and their Principles Imperially Composed* (*Yuzhi Shuli Jingyun* 御製數理精蘊, 1723), a compilation integrating Chinese and Western mathematical writings and discursive forms.[45] Naturally, representing integer sequences of higher order by geometrical shapes has its cognitive limits. Hua Hengfang admitted this, but doomed diagrams as unnecessary tools for making apparent mathematical "principles" once algebra as an all-encompassing technique is used to express and manipulate symbols and formulas :

> Among the ancient mathematical books, none of them draws many charts. In the *Nine Chapters*, there is only the diagram of base and height [in a right-angled triangle]. By eliminating the blue [surfaces] and inserting the red ones it is explained that the sum of the two squares of base and height is equal to the square on the hypotenuse. For all the other procedures no [other ancient mathematical book] resorts to a diagram to elucidate it. […] The problem is that if one uses diagrams to make calculation principles apparent, one can only do so up to solids, yet, this is the end. For higher dimensions, one cannot draw diagrams. For that which diagrams cannot make evident, mathematicians have additionally invented algebra. Among all pieces of segments there is none that cannot be elucidated by a mathematical formula. Therefore, what in ancient times could not be elucidated without a diagram, today can be done so without necessarily using a diagram. What in ancient times no diagram could be drawn for, today can be expressed by algebraic formulas, replacing the function of diagrams. In my mathematical books I never draw diagrams, the reason being also that I use algebraic formulas.[46]

Besides the potential possibility or impossibility to express "principles" verbally or visually, there is another important connection between diagrams and text with respect to philosophical aspects to be found in mathematical writings. It relates to

43. Cf. *supra*, n. 17.

44. Cf. Andrea BRÉARD, "On the Transmission of Mathematical Knowledge in Versified Form in China", in Alain BERNARD, Christine PROUST, ed., *Scientific Sources and Teaching Contexts Throughout History*, Dordrecht, Springer, 2014, p. 164-166.

45. Commissioned by the Kangxi 康熙 emperor (r. 1661-1722), the *Essence* represents a synthesis of Chinese and Western mathematical knowledge. Cf. Catherine JAMI, "The *Yuzhi Shuli Jingyun* (1723) and Mathematics during the Kangxi Reign (1662-1722)", in Yang CUIHUA, Huang YILONG, ed., *Jindai Zhongguo Kejishi Lunwenji* 近代中國科技史論文集 (Science and Technology in Modern China), Taipei, Zhongyang yanjiuyuan jindai shi yanjiusuo guoli Qinghua daxue lishi yanjiusuo, 1991, p. 155-172. Illustrations related to piles of discrete objects therein show distinctly Euclidean features, unknown to the Chinese mathematical literate before the arrival of the Jesuits and the first (partial) translation of Euclid's *Elements* in 1607 : points in the figures are named and referred to in the text which justifies the algorithm to calculate the number of discrete cubes in precisely the depicted stack. Otherwise no obvious graphical pun to European sources, in particular to Clavius' work, can be recognized here. Cf. HAN Qi, "Kangxi Shidai Xifang Shuxue Zai Gongting de Zhuanbo-yi An Duohe 'Suanfa Zuanyao Zonggang' de Bianzuan Wei Lie 康熙时代西方数学在宫廷的传播—以安多和《算法纂要总纲》的编纂为例" (The Circulation of Western Mathematics at the Court during the Kangxi Period - A Case Study of the Compilation of the *Suanfa Zuanyao Zonggang* by Antoine THOMAS), *Ziran Kexueshi Yanjiu*, 22, 2 (2003), p. 149 for the possibility that scroll 16 of Antoine Thomas' manuscript was a source for the chapter on "Heaps of piles" (*dui duo* 堆垛) in the *Essence*, but the version preserved in Japan does not contain any illustrations.

46. Translated according to the essay "About the Question of Being Able to Write Books while Learning Mathematics" (*Lun Suanxue Zhong Keyi Zhushu zhi Shi* 論算學中可以著書之事), in HUA Hengfang, *Xue Suan Bitan*, 12.9a-12.9b (punctuation is mine).

textual and visual practices of inductive argumentation. I have shown that pebble diagrams, by depicting a geometric interpretation of sequences of natural numbers, form an epistemological unity with the accompanying text in Wang Lai's essay *The Mathematical Principles of Sequential Combinations* (*Dijian Shuli* 遞兼數理).[47] Both are persuasive representations of a procedural aspect that establishes a recursive relationship between the mathematical objects that are either depicted or constructed rhetorically.[48] Each item is produced out of the previous result through specific operations. By decomposing geometric figures into layers of unitary elements, the structure of the successive diagrams refers to the patterns of recursive algorithms and inductive arguments.

In contrast to Wang Lai, explicit meta-discursive elements are entirely absent in the case of Li Shanlan. The analogy between diagram and text rests entirely on structures : Li shows the first few numbers in the first few diagonals of a triangular table with unitary pebbles geometrically arranged, and, for each of these first diagonals, gives the procedure that calculates the sum of the first n terms. As for the latter, the reader is asked to infer inductively by analogy (*lei tui* 類推) the general summation algorithm valid for all diagonals, whereas for the former, the invitation is visually extended : may the reader continue the pattern by piling up further unitary building blocks. Figure 2 (left), for example shows in the right column the squares of 1, 2, 3 and 4, the left column the squares of 1, 1+2, 1+2+3 and 1+2+3+4. These are the first numbers in the second and third diagonal (from top) in the corresponding triangular table (see figure 2 on the right, cells colored in red and blue). The text which follows then gives the corresponding algorithms for calculating the sums of $1^2+2^2+3^2+4^2+\ldots+n^2$, $1+(1+2)^2+(1+2+3)^2+(1+2+3+4)^2+\ldots+(1+2+\ldots+n)^2$ as well as the algorithms for the next two series before stating that "beyond these one can proceed by analogical extension" the general procedure for the sums of the first n cells in any diagonal of the triangle.

47. A discussion and complete translation of the text by Wang Lai can be found in Andrea Bréard, "Inductive Arguments in the Midst of Smoke : 'Proving' Rhetorically and Visually that Algorithms Work", in Martin Hofmann, Joachim Kurtz, Ari Levine, ed., *Powerful Arguments : Standards of Validity in Late Imperial China*, Leiden, Brill, 2020, p. 234-276.

48. Wang distinguishes himself by the fact that he connects for the first time in China the summation of finite series to combinatorial considerations and represents the number of possibilities to choose k objects among n by diagrams for figured numbers. They are exactly the same diagrams as the ones found for the first triangle in Li Shanlan's *Comparable Categories*.

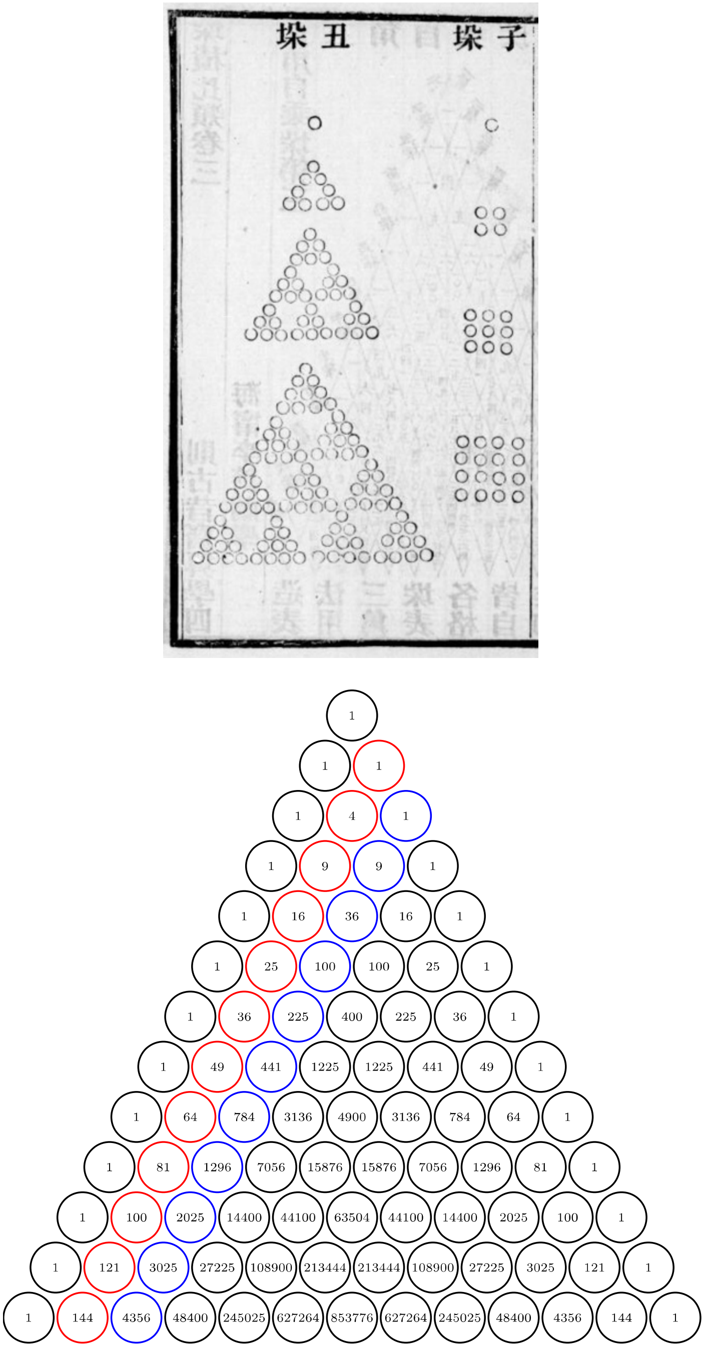
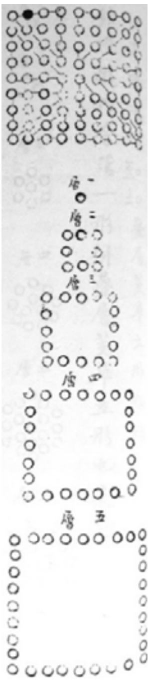


Figure 2 : Piles of squared triangular [numbers] (LI Shanlan, *Duoji Bilei,* 1867, *juan* 3, p. 1b) and corresponding triangle.

A slightly earlier author and friend of Li Shanlan, Dai Xu 戴煦 (1805-1850), in his 1844 commentary to the *Jade Mirror of Four Unknowns* has equally contributed to the field of "discrete accumulations" by providing diagrams of the mathematical objects involved. Yet, he does neither figure in the above list (table 1) nor is he referred to by Li Shanlan in spite of the fact that Dai might have been a source of inspiration for him.[49] Dai indeed proposes an approach to Zhu Shijie's series by diagrams similar to the ones found in Li Shanlan's work. An example from Dai's manuscript allows to illustrate the role which diagrams could play in depicting justifications of procedural patterns even beyond an inductive scheme.



Considering the sequence of even numbers 2i ($i = 1, \dots n$), each taken four times : $4 \cdot 2i$, Dai visualizes the sequence of these terms for $n = 4$ by "empty" squares with each side composed of an odd number $2i + 1$ of pebbles. The sum of these empty squares, arranged around an initial unity at its center, visually gives a "full" square where each side is equal to $2n + 1$. One might thus conjecture a first formula of summation :

$$1 + \sum_{i=1}^{n} 4 \cdot 2i = (2n + 1)^2$$

or[50] :

$$1 + 8\sum_{i=1}^{n} i = (2n + 1)^2$$

By observing in detail the "full" square on top, one recognizes a second "formula" : the square is composed of six triangles with n, a big triangle with $n + 1$ and a small triangle with $n - 1$ units at its base and as its height. One thus also recognizes the following identity :

$$1\sum_{i=1}^{n+1} i + 6\sum_{i=1}^{n} i + 1\sum_{i=1}^{n-1} i = (2n + 1)^2$$

It is precisely this last type of "formula" that Li Shanlan establishes (in procedural language) for the sums in the diagonals in his generalized arithmetic triangles. The "principles" of mathematical procedures, provided without a discourse of justification, are thus visualized in the diagrams. If extended to the general case, i.e. when the

49. The manuscript of Dai Xu's commentary, dated from 1844, contains a commentary annexed (*fu* 坿) to the first scroll which is entirely dedicated to "tables for heaps of piles" (*dui duo biao* 堆垛表). For a complete translation, cf. A. BRÉARD, *Re-Kreation eines mathematischen Konzeptes im chinesischen Diskurs*, p. 438-451.

50. This same link between sums of natural numbers (leading to triangular numbers) and square numbers is attested in ancient Greek sources : cf. DIOFANTO, *De polygonis numeris*, introduzione, testo critico, traduzione italiana e commento di Fabio Acerbi, Pisa, Roma, Fabrizio Serra, 2011, p. 45.

pattern has been recognized, these diagrams can play the role of a proof without words. That such was indeed the intension of the authors, we can only argue on the ground of statements such as the one quoted above by Zhu Shijie, claiming that “the principles are of evident nature” if one studies the diagrams. The massive presence of pebble diagrams in Dai Xu’s and Li Shanlan’s work would certainly confirm the importance attached to the epistemological function of visualization as a tool for justification (and discovery).

Another kind of stacked numbers systematically presented and strategically placed within the mathematical writings relevant to “discrete accumulations” are the numerical tables in triangular shape (cf. fig. 3). These triangular diagrams display a high degree of freedom as concerns the direction with which they were to be read : the lines connect the cells with the neighboring cells above, below and on their sides (if there are any). Yet, what seems like an interpretative ambiguity is productive mathematically.[51] Read horizontally or diagonally, the tables do not allow to produce the same mathematical meaning : binomial coefficients in one way, arithmetic series in the other, the manifold “uses” of arithmetic triangles have been developed systematically in the West by Pascal in his *Traité du triangle arithmétique* (1667). Pascal, as well as Zhu Shijie and Li Shanlan, place the diagram before or right at the beginning of their book, thereby making manifest its crucial necessity as a foundation for the methods to follow.

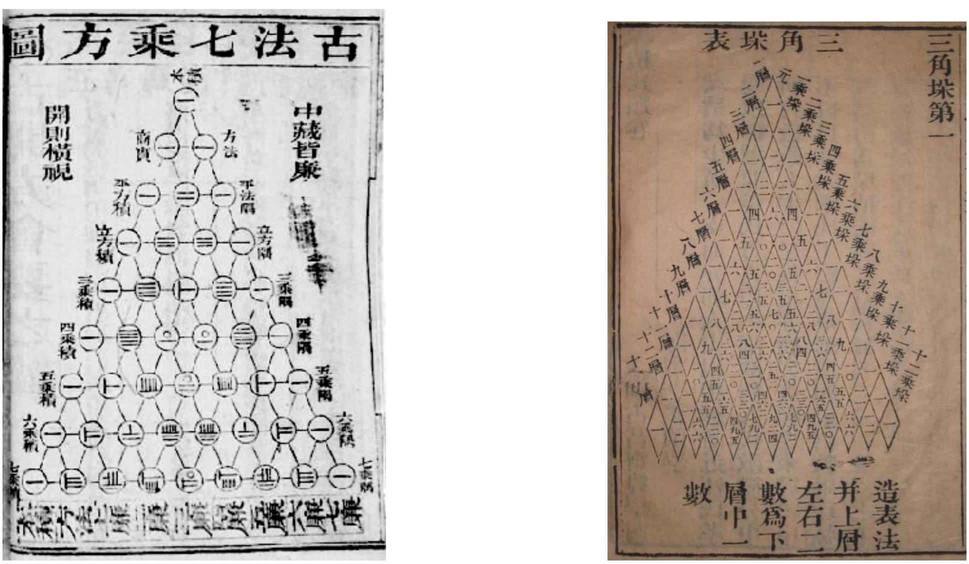


Figure 3 : The “Pascal Triangle” in ZHU Shijie, *Siyuan Yujian*,
plate 1 of the preliminary diagrams (left) and p. 1b following Li Shanlan’s preface (right).

51. For comparison, cf. Emily GROSHOLZ, *Representation and Productive Ambiguity in Mathematics and the Sciences*, Oxford, Oxford University Press, 2007 on the cognitive importance of ambiguity in diagrammatic representations in Descartes’ *Geometry*.

Other arrangements of such triangular tables are also to be found in other Chinese mathematical texts. Among the authors listed in table 1 and mentioned by Li Shanlan in his preface as one of his predecessors, for example, is Dong Youcheng. Compared to Zhu Shijie, Dong does not provide any new summation "formulas" but he does change the layout of integer sequences in a diagram, arranging them into rectangular tables.[52] Others, like Fu Jiuyuan, even integrate verbally the combinatorial interpretation of arithmetic series into the diagram of the arithmetic triangle. By arranging differently the cells for each term (cf. fig. 4), Fu indicates horizontally and to the right of each cell the combinatorial meaning of each numerical value "*k* of *n*" (*n* 之 *k*), which is equivalent to the binomial coefficients $\binom{n}{k}$.

As in the pebble diagrams, a general pattern of generation of the content of the represented cells can be induced from the limited number of cells depicted. The structure of and the relation between the numbers involved are therefore made apparent, even if they are not accompanied by any conceptual discourse.

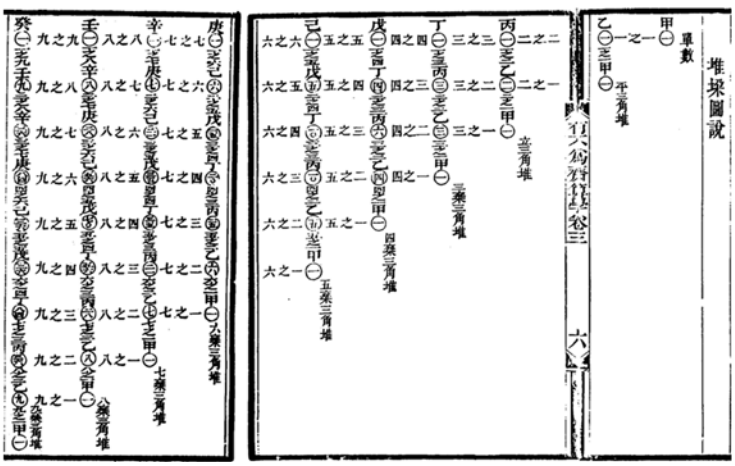


Figure 4. Fu Jiuyuan 傅九淵, Illustrated Explanation of Heaps of Piles (*Duiduo Tushuo* 堆垛圖說), 1888/1889.[53]

52. For Dong's use of diagrams, cf. A. BRÉARD, "What Diagrams Argue in Late Imperial Chinese Combinatorial Texts", p. 259-261.

53. FU Jiuyuan 傅九淵, "Chaocha shujie 超差術解" (*Explanation of the Procedure of Divided Differences*) in ID., *You Bu Wei Zhai Suanxue* 有不為齋算學 (Mathematical Learning from the Studio where Certain Things are not Done), vol. 1, Dehua Li Shi [= Li Shengduo] Muxi Xuan Congshu 德化李氏[=李盛鐸]木犀軒叢書 ed., 1887/1888 [Reprint *Congshu Jicheng Xu Bian* 叢書集成續編 (Continuation of the Collected Collectanea), "*Ziran Kexue Lei*", vol. 76, p. 358-365, Taipei, Xinwen feng, 1989], here p. 3.6-3.7a.

CONCLUSION

Coherent traces of epistemological assumptions about the "principles" of mathematical objects and procedures have been shown to exist in the so-called field of "discrete accumulations", or what in the West is known as figured numbers. A system of classification and of reasoning in textual and diagrammatic form evolved historically for these mathematical objects until the turn of the 20^{th} century, influenced by concepts that were common in philosophical discussions about ways of knowing outside of the mathematical domain of inquiry. While in ancient Greece questions on foundations of mathematics were treated by Plato and Aristotle within the framework of metaphysical reflections, no explicit discussion of the nature of mathematics, its objects and tools, is recorded in the transmitted sources from China. Yet, a philosophical interest in cognition, representation and justification is evidenced in normative narratives about mathematical objects and their visualization in the mathematical literature itself. The imagined dialogue between Russell and Zhu Shijie reflects these two kinds of philosophical narrative with respect to numbers : while Russell discusses explicitly and spontaneously his own ideas about the nature of numbers, Zhu Shijie responds through quotations from mathematical texts, refers to commonplace diagrams or phrases, and thus implies philosophical content by letting the reader engage with these elements.