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Bruno-Laurent Moschetto and Frédéric Teulon

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Article abstract

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An anti-takeover strategy by limitation of voting rights: A model and a numerical approach

Une stratégie anti-OPA par limitation des droits de vote : une approche théorique et numérique

Estrategia anti-OPA limitando los derechos de voto: un enfoque teórico y numérico

BRUNO-LAURENT MOSCHETTO Criisea, university of Picardie Jules Verne, France

RÉSUMÉ

Cet article étudie une stratégie de défense anti-OPA fondé sur le principe de limitation des droits de vote, mis en place par un petit groupe d'actionnaire qui détient le pouvoir au sein d'une société cotée. L'article propose de modéliser l'impact de la limitation des droits de vote sur l'environnement puis s'intéresse à l'optimisation du modèle de défense qui doit protéger la firme tout en la maintenant attractive pour les investisseurs non hostiles. Le modèle théorique qui repose sur l'exploitation d'un *seuil de limitation* et d'un *coefficient de réduction* est illustré par différentes simulations numériques.

Mots clés : Stratégies anti-OPA, Contrôle de la firme, Gouvernance, Enracinement des dirigeants, Limitation des droits de vote

FRÉDÉRIC TEULON Ipag Lab Ipag Business School, France

ABSTRACT

This study develops a new trade-off view of corporate governance from an examination of rules that limit voting rights as a defensive measure against a hostile takeover attempt. The theoretical framework concerns a listed company, the capital of which is mainly detained by atomistic shareholders and the power of which is in the hands of a minority shareholders, the hard core. The latter wants to block any hostile takeover and constructs a device based on two parameters allowing it to act on the limitation of the voting rights: *a threshold* and a *scale-down coefficient*.

Keywords: Anti-takeover amendments, Control-enhancing mechanisms, Corporate governance, Management entrenchment, One-share one-vote principle

RESUMEN

Este artículo estudia una estrategia de defensa anti-OPA basada en el principio de limitación de los derechos de voto, creado por un pequeño grupo de accionistas que detenta el poder en una sociedad cotizada. El documento propone modelar el impacto de la limitación del derecho de voto en el medio ambiente y se centra en la optimización del modelo de defensa para proteger a la empresa manteniéndolo atractivo para los inversores no hostiles. El modelo teórico basado en la explotación de un umbral de limitación y un coeficiente de reducción es ilustrado con varias simulaciones numéricas.

Palabras Claves: Estrategias anti-OPA, Control de la empresa, Gobierno, Enraizamiento de los líderes, Limitación de los derechos de voto.

There is strong opposition in the industry to any sort of limitation of voting rights. According to Arnold (2013), this is true even when such limitation is framed not as a reduction, but in fact as some sort of an increase of voting or cash-flow rights. In this study, we choose to examine a specific defense strategy, the linear rule defense mechanism, based on the limitation principle. This choice is justified by the fact that many of the control-enhancing mechanisms (CEMs) that firms employ can be seen as special cases of the linear voting rule that this study introduces. We know that this choice is not necessarily widespread in this research field, but we mainly argue for it from a theoretical point of view.

There is a vast amount of literature on takeover bids and on the use of CEMs, as Adams and Ferreira (2008) show, but paradoxically, a number of issues have not yet been sufficiently addressed. Therefore, there are still no formal models with the optimal combination of parameters available to managers of a company seeking to protect themselves from a hostile takeover bid without significantly damaging the interests of shareholders. Although it remains rather heterogeneous on the subject, the literature concerning the protection against takeover bids can be divided into three broad categories. The first is interested in the nature of the protection and its efficiency; the second approaches the problem from the point of view of governance; and the third is especially interested in the impact of the defensive measures on the firm and its environment.

As regards the nature of the protection and its supposed efficiency, there seems to be a gap between theory and empirical studies. In addition, the few existing models concern only a few of the facets of bids, which limits their scope and applicability. One of the earliest articles on the subject is that of Shleifer and Vishny (1986b), who model greenmail interactions. A few years later, Bagnoli, Gordon, and Lipman (1989) build a model whereby the manager is prompted to give information about the firm's value through share buying.



From the 1990s, the proposed defense strategies, both theoretical and practical, have multiplied. Austen-Smith and O'Brien (1992) are interested in defensive operations that require shareholder approval, whereas Molin (1996) analyses the impact of changes in the control threshold. He shows that poison pills are optimal under parameter configurations that imply a high *a priori* takeover probability, consistent with the empirical evidence. More recently, the research on takeover bids and related defense systems began to take a quantitative turn. For example Goldman and Qian (2005) calculate the optimal number of toeholds a bidder should acquire before launching a takeover bid and offer an explanation for why raiders do not acquire the maximum possible number of toeholds prior to announcing a takeover bid.

As regards the protection against takeover bids from the point of view of governance, the classical notions of governance and efficiency are in the center of the debate. In most studies that deal with restructuring and control issues (Jensen, 1988; Tirole, 2006), takeovers are generally presented as a useful tool for preventing the management from taking root and as the only external control tool in businesses with dispersed ownership. However, an increase in the number of takeovers seems to push the managers in place to adopt short-term strategies and, at times, may also call the national control of an industry into question. At the same time, most management teams remain opposed to any sudden change in control (Shleifer and Vishny, 1986a), and consequently, the adoption of defensive measures against hostile takeovers appears to be justified. While defense strategies today are strictly controlled, those based on limitation of voting rights are reputedly effective, albeit often criticized by finance scholars. In effect, it is generally considered that voting rights proportionate to the distribution of share capital usually lead to better governance (Bebchuk, Cohen, and Ferrell, 2009; Grossman and Hart, 1988; Harris and Raviv, 1988; Jensen, 1986) and enhanced managerial discipline (Armstrong, Lange, and Woo, 1994; DeAngelo and Rice, 1983; Johnson and Rao, 1997). More recently, the study of Core, Guay, and Rusticus (2006) demonstrates that firms, in equilibrium, employ governance regimes that enable them to operate the most profitably and that once the markets learn about this, governance becomes irrelevant. Finally, At, Burkart and Lee (2011) consider an investor's efficiency relative to the incumbent owners'.

Finally, as regards to the impact of defense systems on the firm and its environment, many works have found evidence that strong shareholder orientation is associated with high stock returns. The evaluation of the costs and advantages of anti-takeover protection measures is tricky (Bebchuk and Cohen, 2005). Turk, Goh, and Ybarra (2007) suggest that if anti-takeover measures do not lead to a downward revision of the profit forecast at the time they are adopted, they are doubtlessly a response to their prior decline. In addition, it would appear that there is often a significant reaction by the finance market to the adoption of such measures. This is also true with regard to finance analysts' assessment of the firm in question (Johnson and Rao, 1997). Grossman and Hart's study (1980b) is generally considered as a seminal theoretical work in this field. The baseline result that free-riding prevents value-increasing takeovers from succeeding has been resolved through three potential channels in the literature. 1. Grossman and Hart (1980a) posit that bidders can threaten to punish non-tendering shareholders ex post by diluting their stake. They analyze exclusionary devices that can be built into the corporate charter to overcome this free-rider problem. 2. Shleifer and Vishny (1986a) examine the use of toeholds. They consider a setting where multiple shareholders have endogenous conflicts of interest depending on the size of their stake. 3. Bagnoli and Lipman (1988) question the original assumption that no shareholder is pivotal. Our model belongs to the third strand of the literature because the hard core knows that it can give the bidder a hard time by refusing to sell. However, our study considers that the adoption of over-the-top defensive measures is unquestionably perceived as negative by the market, resulting in investors backing off.

We know that institutional investors and specifically, mutual funds, hold a large fraction of US corporations' equity (Wermers, 1999). Moreover, the prevalence of pure index funds has grown rapidly, reaching 15% by the end of the 1990s (Cremers and Petajisto, 2009). It is therefore reasonable to assume that the cost of limiting shareholder voting is constant at 0 for some investors because they have no option but to invest in the firm's stock. In our study, we clearly suppose that such costs exist in particular because of the loss of liquidity and the decreased access to capital.

From a theoretical point of view, this study seeks to remain completely neutral on the topic of the potential destruction of value for shareholders. If our arguments appear to favor the defense of the shareholders in place, this is simply because of our search for clarity and our will to model the means of the defense and not those of the attack. Consequently the takeover is treating as a constraint (the takeover must be avoided for sure), and while the impact on the price is optimized. The model is built on an arbitrage between conflicting goals (efficiency of the device and potential destruction of value).

Our study contributes to the literature in several ways. First, our analysis suggests the quantification of the links between the presence of an anti-takeover bid device and the value of the firm. Second, this study demonstrates that the members of the hard core can calculate the necessary ownership threshold to protect themselves from a takeover threat. Third, our study explores the resources available to management teams or family shareholders who wish to reduce the risk of a takeover or, conversely, the resources available to an investor who wishes to take control of a firm with a united but not predominant hard core. It analyzes the performance and restrictions of the linear limitation of voting rights, designed to counter the ambitions of a single hostile investor.

The rest of the paper is organized as follows. Section 2 presents the framework and the defense strategy. Section 3 highlights the impact of the device on the control of the firm, on the price of the share, and on the attractiveness of

the potential takeover bid. Section 4 describes the optimization program, which determines the best adjustment of the measure. Section 5 describes a numeral calculation. Section 6 presents the conclusions and outlines the implications for future research. The appendix presents all the mathematical proofs.

The study framework

The study concerns a listed company, the capital of which is comprised of *S* shares, considered as constant (Betton, Eckbo, and Thorburn, 2008; Calgano and Falconieri, 2013). The firm may be the target of a hostile takeover in the near future. Three types of actors are involved: 1. a large number of passive and fragmented shareholders, hereafter referred to as *the small shareholders*; 2. a small group of united but minority shareholders who refuse any type of takeover bids and who form *the hard core*; and 3. a single *hostile investor* who is the initiator of the takeover bid.

For simplicity, the costs and benefits to the hostile investor are not modeled. We suppose that the shareholders of the hard core do not need to evaluate them and can make a decision on the anti-takeover measure without a certain expectation about the hostile investor's intention after he/she becomes a majority voter.

THE SMALL SHAREHOLDERS

Collectively, small shareholders hold the majority of shares, but no one shareholder holds a significant number of shares.

Although it establishes a simplification, we hold two postulates: 1. their attitude is guided by uniquely pecuniary considerations, 2. they never attend the general assemblies. According to the literature, when a takeover bid starts, the small shareholders split into two groups. The members of the first prefer to sell their shares immediately to the highest bidder, while those of the second prefer to wait, wondering about whether the sale can be postponed. Theoretically, a takeover gives small shareholders the possibility to add value by selling their shares to the potential investor at a high price. In practice, however, it is better for them not to sell their shares too quickly in the hope of a higher bid.¹ The small shareholders will sell only if they consider that the price offered during the takeover is higher than the future value of their shares and if they believe that the probability of a higher bid is very unlikely. In this case, we postulate that the proportion of the small shareholders avid to sell their shares immediately is directly connected to the difference between the market price of the shares and the price proposed for the takeover.²

The hard core

The hard core is made up of *N* shareholders who systematically take part in the general assemblies. They are classified from 1 to *N* in decreasing order of importance according to the number of shares they hold, where shareholder *i* holds l_i shares. Together, the hard core shareholders hold S_{he} shares, which amount to less than half the number of shares issued. Our study remains in a deliberately limited framework, which, in particular, precludes any change to the shareholder structure within the hard core during the takeover period. The members of the hard core cannot either exchange shares among them or buy shares to the small shareholders. This limitation makes the hard core vulnerable to the arrival of a new hostile investor, who would be tempted to purchase a massive number of shares during a takeover bid.

The hostile investor

The hostile investor, characterized by the index 0, acts alone and wishes to launch a speculative takeover of the firm. Initially, he/she holds no shares in the target firm. At the end of the takeover, the objective of the hostile investor is to control the firm. For that purpose, he/she hopes to hold l_o shares, more than the number of shares held by the hard core.³ This potential amount of purchase is directly bound to the price P_o announced by the hostile investor during the takeover bid.⁴

THE DEFENSE STRATEGY

The company has an anti-takeover defense measure that limits voting rights. This type of defense is closely tied to the legal framework under which it operates⁵ and is minimalist from a legal viewpoint. The limitation measure is considered legal as long as it applies to all of the shareholders without exception.

For the members of the hard core, who want to protect themselves, the protective system consists of fixing a number of reference shares, called the *threshold* and denoted as δ . If a shareholder holds a number of shares lower than or equal to this threshold, each share is linked to a voting right.⁶ This is the case for all the small shareholders, and we assume that this is also true for certain members of the hard core. Conversely, shareholders are concerned by the measures if they own more than δ shares. Above this threshold, the

^{1.} Grossman and Hart (1980a); Holmström and Nalebuff (1992).

^{2..} In fact, the attitude of the small shareholders is uncertain. The majority of them are probably going to react rationally (the more the proposed price is raised, the more the sales of shares will be important). Nevertheless, a minority of them will be probably less reactive. Two reasons ca be moved forward: the will of favouring the strategic approach of the investment to the detriment of the financial interest and the heaviness of administrative procedures bound to the sale when this one concerns a very low number of shares. For purposes of simplification, we ignore voluntary these phenomena and consider the attitude of the small shareholders as being only guided by the prices.

^{3.} A successful offer requires that the bidder attracts at least 50 percent of the firm's voting rights in the general assemblies. This is the condition needed to gain control as we assume that the hard core coalition will not weaken.

^{4.} The price P_0 is an exogenous variable.

^{5.} La Porta, Lopez-de-Silanes, Schleifer, and Vishny (1998).

^{6.} According to the one-share one-vote principle.

number of voting rights concerned is multiplied by a *scale-down coefficient*, also fixed by the firm, noted as γ and bounded⁷ between 0 and 1.

 δ is a threshold – concerning the numbers of shares possessed by each shareholder – below which each share is linked to one vote and above which each share gives less than one vote.

This situation concerns some of the members of the hard core, and implicitly, we suppose that this is also the case for the hostile investor (the model does not present an interest if this number is lower than or equal to δ).

We note *K* as the number of shareholders of the hard core, whose number of shares exceeds δ , and S_{hc}^f as the total number of the shares of the hard core that is not subject to limitation. By definition, *K* is always between 0 and *N* and the inequality $S_{hc}^f \leq S_{hc}$ is always confirmed. We note that both *K* and S_{hc}^f depend on δ . For the first variable, there are discontinuities from thresholds, while for the second variable, the evolution is continuous. If $\delta \geq l_1$, nobody in the hard core is concerned by the limitation: K = 0 and $S_{hc}^f = S_{hc}$. If $l_2 \leq \delta < l_p$ only the first shareholder is concerned by the limitation: K = 1 and $S_{hc}^f = S_{hc} - (l_1 - \delta) = S_{hc} + \delta - l_1$. If $l_3 \leq \delta < l_2$, only the two first shareholders are concerned by the limitation: K = 2 and $S_{hc}^f = S_{hc} - (l_1 - \delta) - (l_2 - \delta) = S_{hc} + 2\delta - l_1 - l_2$. Let us write a relation of recurrence:

If
$$l_{k+1} \le \delta < l_k$$
 then $K = k$ and

$$S_{hc}^f = S_{hc} + K\delta - \sum_{i=1}^K l_i = K\delta + \sum_{i=K+1}^N l_i$$
(1)

After applying the measure, the total number of voting rights that shareholder *i* holds in the general assembly, written as VR_i , is the sum of his/her unlimited and proportional rights f_i om which he/she is likely to benefit. If $l_i \leq \delta$, then $VR_i = l_i$, and if $l_i > \delta$, then $VR_i = \delta + \gamma(l_i - \delta)$ or $VR_i = \gamma l_i + (1 - \gamma)\delta$. Thus, if at least one shareholder owns a number of shares above the threshold δ , the total number of voting rights that can be used in the general assembly is below the number of shares held by the shareholders present.⁸ This number is denoted as VR and is obtained by adding the number of unlimited shares and all the reduced rights. According to appendix A,

$$VR = \gamma (l_0 + S_{bc}) + (1 - \gamma)(\delta + S_{bc}^{f})$$
(2)

The impacts of the device

The impact of the device on the control of the firm

The hard core shareholders initially hold the power and wish to retain it without any division. In this study, how this power evolves is thus a central issue, and the main variable characterizing this power is the percentage of voting rights held by the different shareholders during the general assemblies. We term this percentage as *PVR*_i relative to shareholder *i*. The effectiveness of the limitation of voting rights measure can be evaluated in relation to the percentage of voting rights that the hostile investor manages to obtain, in other words,

$$PVR_{0} = \frac{VR_{0}}{VR} = \frac{\gamma l_{0} + (1 - \gamma)\delta}{\gamma (l_{0} + S_{hc}) + (1 - \gamma)(\delta + S_{hc}^{f})}$$
(3)

This percentage depends seemingly on the values of $(\delta, \gamma, l_0, S_{hc}, S_{hc}^f)$, but according to equation (1) the only parameters relevant are: 1/ the variables $(\delta, \gamma, l_0, S_{hc})$ and 2/ the distribution of the S_{hc} shares between the different members of the hard core.⁹

The shareholders of the hard core and the hostile investor are the principal decision-makers in the model. In order to keep their control of the firm, the members of the hard core determine the voting rule for the shareholder assembly in their favor by under-weighing the votes of large shareholders. They determine, first of all, the maximum percentage of voting rights that the hostile investor can obtain.¹⁰ In the rest of this paper, this maximum is designated by M. Then, the members of the hard core estimate the number of shares (l_0) that the hostile investor is potentially capable of buying considering the level (P_0) of his offer. According to the value of S_{hc} and of the distribution of these shares inside the hard core, they then configure the defense system by determining the values of δ and γ . Because these values are fixed, those of S_{hc}^{f} and K deduct.

The values of δ and γ must be chosen carefully so that the device is effective against the hostile investor but discreet enough not to scare off the small shareholders. In theory, M is bounded between 0 and 1, but the model has interest only for values of M lower than or equal to $\frac{1}{2}$. As the hostile investor acts alone, this target can always be achieved. To this end, it is sufficient to say $\delta < S_{hc}$ and $\gamma = 0$. Such a choice systematically leaves the hostile investor in a weaker position than the traditional hard core, which remains united. The measure can thus always counter a single hostile investor. In practice, the lower the value of M, the more the hard core is protected. Two values are particularly susceptible, namely $M = \frac{1}{2}$ and $M = \frac{1}{3}$, which correspond, respectively, to the majority during ordinary general assemblies and

^{7.} The value $\gamma = 1$ corresponds to no reduction of the voting rights theoretically subjected to limitation; this cancels the interest of the device. In contrast, for $\gamma = 1$ all the shares which a shareholder possesses beyond the threshold are cancelled.

^{8.} This is the case at the end of the purchase of the shares as the hostile investor is always assumed to go beyond the threshold.

^{9.} This distribution is fixed in the article; it determines the value of K and thus, that of S_{hc}^{J} .

^{10.} This number must be lower than or equal to 50%; it ensures that the quality of the protection also influences the attractiveness of the firm.

to the blocking minority.¹¹ The hard core may also decide to fix a value for *M* below $\frac{1}{3}$ because the weaker the value of *M*, the more dissuasive the device.

A priori, the hostile investor has no sure knowledge of the values of δ and γ He/she is, however, always capable of anticipating them with a degree of uncertainty. According to his/her forecasts, he/she can decide to maintain his/her offer (P_0) at his/her initial level. He/she can also increase his/ her offer to try to increase the proportion of minority shareholders avid to sell their shares at once and thus, to increase the value of (l_0). Finally, he/she can decide to withdraw his/ her offer, considering that he/she has few opportunities to achieve his/her goal.

THE IMPACT OF THE DEVICE ON THE PRICE OF THE SHARE

To counter the hostile investor effectively, the threshold value of δ needs to be as low as possible and the associated value of γ should be as close to 0 as possible. However, if the company wishes to retain the shareholders' confidence and to continue to attract *non-hostile* investors effectively, the measure needs to be as subtle as possible. Indeed, activating the system too abruptly risks making the small shareholders wary as the neutralization process for voting rights is viewed as damaging to good governance. This means that the threshold value of δ should be as large as possible and the associated value of γ , a factor keenly observed by the market, as close to 1 as possible. To be more precise, we acknowledge that the activation of the defense system might be rejected by the small shareholders, which would generate significant sales of shares. These sales have a negative effect

on the theoretical price of the share (P_t) , which is the price of the firm in the absence of the device. This effect increases as δ becomes lower and γ becomes closer to 0.

But the sensibility of the price according to these two parameters is not linear. Indeed, if γ is close to 1, the negative impact of the threshold is smaller, sometimes insignificant, because whatever the value of the threshold, very few of the shares subjected to limitation are finally reduced. In contrast, if γ is close to 0, the negative impact of the device on the price can be very important. In the rest of the study, we shall call (P_m) the market price, that is, the price observed in the presence of the defense device. To integrate the aforesaid negative effect, we postulate the following link between P_m and P_i :

$$P_{m} = \frac{P_{t}}{Ln(P_{t})} \times Ln \left[aP_{t} \frac{\delta}{S} + (1-a)P_{t} \left(1 - \left(1 - \frac{\delta}{S} \right) (1-\gamma) \right) \right]$$
(4)

In this equation, *a* is a variable of the model, a coefficient between 0 and 1 that strengthens or puts into perspective the weight of δ compared with that of γ . So, for an *a* close to 1, the impact of the device on the price is essentially due to the parameter δ . In contrast, for an *a* close to 0, it is the impact of γ that becomes essential. For illustrative purposes, let us assign outstanding values to the parameters δ and γ . For $\delta = S$, the equation becomes $P_m = P_t$ (the maximum level of the threshold cancels all the efficiency of the device). For $\delta = 0$, the equation becomes $P_m = [P_t / Ln(P_t)] \times Ln[(1-a)\gamma P_t]$. A value of γ close to 1 implies a value of P_m rather close to P_q , even though the threshold is equal to 0. On the other hand,



11. The value $M = \frac{1}{2}$ involves $l_0 < \frac{1}{2}$, and the hostile investor cannot become the majority on his own during general assemblies. The value $M = \frac{1}{3}$ involves $l_0 < \frac{1}{3}$.

a low value of γ significantly increases the efficiency of the device and therefore, corresponds to a low value of P_m . These results are stressed or put into perspective by the values of a. To demonstrate, graph 1 gives the values of P_m as a percentage of the value of P_p , with the values of S and a arbitrarily fixed (S = 1000000; a = 0.1). The value of δ evolves into [0,S], and γ successively takes six different values between 0.02 and 1.

Of course, the evolution of the share's market price is not ineffective on the wealth of the shareholders of the hard core. Rationally, they will choose values for δ and γ that maximize their assets (i.e., a value for P_m that is as high as possible).

THE IMPACT OF THE DEVICE ON THE ATTRACTIVENESS OF THE TAKEOVER BID

We know that the attitude of the small shareholders is partially uncertain. However, like it is specified in the study framework (in particular through the footnote $n^{\circ}2$), we apply a number of simplifications in order to model more easily the behavior of the small investors.

We consider that in the case of a takeover bid, the small shareholders divide into two groups. The members of the first group sell their shares immediately and make an immediate profit. The members of the second group adopt a wait-and-see approach and sell only if they are convinced that the forward value of their shares will not be superior to the price of the offer. Consequently, the proportion of small shareholders that fall into the first group is directly bound to the level of the offer.

As the offer of the hostile investor rises, the greater the number of small shareholders who are immediate sellers. As a consequence, the number of shares held by the hostile investor l_0 increases as P_0 is important. If we postulate that P_0 can evolve between a minimal price that cannot be legally lower than P_t and a maximum price that is equal to bP_t (with b > 1), then we can write:

$$l_{0} = (S - S_{hc}) \left(\frac{P_{0} - P_{t}}{(b - 1)P_{t}} \right)^{c}$$
(5)

For $P_0 = P_t$ the offer of the hostile investor is not interesting and $l_0 = 0$ whereas for $P_0 = bP_t$, this offer is accepted by all the small shareholders and l_0 is equal to the total number of the shares available in the market,¹² with $l_0 = S - S_{hc}$. The variable *c* is the propensity of the small shareholders to sell their shares (with $c \ge 1$). For c = 1, the proportion of the sellers evolves linearly. For c > 1, this proportion evolves in a convex way. Graph 2 illustrates equation (5) and gives the proportion of the available shares $(S - S_{hc})$ given up immediately by the small shareholders, with the value of b = 1,5 arbitrarily fixed (the parameter *c* successively takes five different values between 1 and 3).

The reduction in the value of the share influences the attractiveness of the hostile investor's offer. The implementation of the device provokes a decrease of the price of the share from P_t to P_m . The offer of the hostile investor becomes all the more attractive as P_m is low, and the difference $P_0 - P_m$.



12. We suppose that the members of the hard core remain united. They are hostile to any takeover bid and on no account would they wish to sell their shares to the hostile investor.

then rises all the more. For the same value of P_t and for the same value of c, a greater number of shareholders decide to sell their shares immediately, which increases the firm's capital l_0 held by the hostile investor. This effect can be modelled by modifying the initial value (c) of the model and by determining the ratio P_{tw} / P_t :

$$l_{0} = (S - S_{hc}) \left(\frac{P_{0} - P_{t}}{(b - 1)P_{t}}\right)^{d} \text{ where } d = c \frac{P_{m}}{P_{t}} + \frac{1}{c} \left(1 - \frac{P_{m}}{P_{t}}\right)$$
(6)

Graph 3 illustrates equation (6) and gives, for c=2, the proportion of the available shares given up at once by the small shareholders, with the value of b=1,5 arbitrarily fixed. If we suppose that the impact of the device is worthless $P_{ii} = P_{i}$, equations (5) and (6) are equivalent. For c = 2, the proportion of the shares sold immediately by the small shareholders is a convex function equal to 0 for the price $P_0 = P_t$ (legal minimum), close to 25% for the price $P_0 = 1,25P_t$, and equal to 100% $(S - S_{hc})$ for the price $P_0 = 1,5P$. If, in contrast, the impact of the device is supposedly bound to the difference $P_0 - P_m$ since the device is activated, the price of the share decreases to P_m , that is, the difference $P_0 - P_m$ strictly superior in $P_0 - P_t$ that serves as a reference to the small shareholders. So, for the same price $P_0 = 1,25P$, the proportion of shares that are sold is close to 30% for $P_m = 0,75P$, close to 40% for $P_m = 0.5P$, and increases until the hypothetical value of P_n is equal to 0. For the lowest values of P_m , the proportion of the shares given up immediately by the small shareholders becomes a concave function.

We also notice that the choice of c=1 corresponds to a value of *d* that is fixed and equal to 1. This choice makes the theoretical discussion much less heavy and is why we shall keep the discussion in section 4 of the paper. In section 5,

other situations are examined and various numerical simulations are proposed.

Optimizing the measure's adjustment

This section presents an optimization program, which determines the best adjustment of the measure in a framework that excludes the conditions for the convening of general assemblies. The measure can be made systematically effective against any hostile takeover but at the risk of sending a negative signal to the finance markets in terms of governance. That is why we look for the best adjustment of the variables on which the members of the hard core can act, which will discourage any hostile investor but will enable the firm to retain the market's confidence in order to attract new minority investors.

THE MAIN ZONES OF APPLICATION OF THE DEVICE

The situations at the limit are unrealistic or without interest. Those in which the values of δ or γ are very low or null send a highly negative message to the market by blocking all possible control of the company. When the value of δ is very close to S or the value of γ is very close to 1, it cancels out the interest of the measure as the situation is the same as in its absence. In fact, the model takes its main interest for the intermediate values of δ and γ . Besides, the implementation of a device that limits voting rights is justified in the face of a really threatening investor (who could hold a number of shares higher than S_{hc} It supposes that the inequality $l_1 < l_0$ is verified. Finally, only the situations for which the hostile investor holds a number of shares above δ present an interest for our study. So, the main zones of application of the device are characterized by three inequalities: $\delta \ge l_N$,



 $\delta < l_0$, and $l_1 < l_0$. In the same way, if from a theoretical point of view the value of *M* can be between 0 and 1, only the situations characterized by $0 < M \le \frac{1}{2}$ are coherent from a practical point of view.¹³ Afterward, we shall suppose that all these conditions are verified.

Depending on the price of the offer (P_0) and the passive shareholders' profile, the hard core members are able to calculate the number of shares that the hostile investor is likely to obtain, which would determine that investor's voting rights. All the other parameters of the problem are known or are fixed by the hard core. According to appendix B, the discussion of inequality $PVR_0 < M$ leads to $l_0 < l_0^{max}$, where l_0^{max} is defined by

$$I_0^{\max} = \left(\frac{M}{1-M}\right) \left(S_{hc} + \left(\frac{1-\gamma}{\gamma}\right) \left(\frac{\delta(M-1)}{M} + S_{hc}^f\right)\right)$$
(7)

It therefore appears that the takeover's success depends on the hostile investor's capacity to obtain a number of shares over or equal to l_0^{\max} . Conversely, the effectiveness of the measure introduced by the hard core depends on its ability to maintain l_0 under the same threshold l_0^{\max} which we will henceforward call the *defensive threshold*.¹⁴ According to equation (1), which gives the expressions of K and S_{hc}^f , it seems necessary to define the value of in each of the relevant intervals of study for the value of δ :

If
$$l_{k+1} \leq \delta < l_k$$
, then

$$l_0^{\max}(k) = \left(\frac{M}{1-M}\right) \left(S_{hc} + \left(\frac{1-\gamma}{\gamma}\right) \left(\frac{\delta(M-1)}{M} + S_{hc} + k\delta - \sum_{i=1}^k l_i\right)\right) (8)$$

THE OPTIMIZATION PROGRAM

The hard core shareholders have to set the values of δ and γ that optimize their defense strategy. The mechanism of defense is set up bit by bit: (i) initially, the hard core chooses arbitrarily the values of the parameters δ and γ ; (ii) the hostile investor decides to launch an attack against the firm and proposes an equal purchase price for all the shares (P_0); (iii) this price determines directly the number of shares which the hostile investor acquires (l_0); (iv) the members of the hard core fix a maximal value for $l_0 l_0^{max}$, the threshold below which they wish to maintain (l_0); (v) by construction, the value of l_0^{max} depend on the structure of the hard core.

Concerning the measure's effectiveness, the hard core shareholders have to determine the values of δ and γ that define a defensive threshold that guarantees the inequality $l_0 < l_0^{\max}$. Concerning at the same time the firm's attractiveness and the maximization of their personal wealth, the hard core shareholders have to determine the values of δ and γ that maximize the value of P_m (the price observed in the presence of the defense device).

As we postulate that the shareholders' structure is fixed within the hard core, the parameters to act on the defensive threshold are thus reduced to the couple (δ, γ) . We therefore have to resolve a program that maximizes $P_m(\delta, \gamma)$ under the constraints $0 \le \delta \le S$ and $0 \le \gamma \le 1$ and under the constraint of efficiency $l_0 < l_0^{\max}$.

The value of the threshold depends on the interval containing δ . The resolution of the program of maximization must be thus driven in each of the intervals¹⁵ $l_{k+1} \le \delta < l_k$ with $k \in \{0, 1, ..., N-1\}$, that is, intervals in which the value l_0^{\max} equals $l_0^{\max}(k)$. We look for the optimum under the constraints of a continuous function that are defined at intervals and what engenders the points of discontinuity in every change of the interval. Theoretically, if the function to be maximized presents local extremums, the calculations of optimization are heavy.¹⁶ In practice, the form of the function $P_m(\delta,\gamma)$ considerably simplifies the approach as long as, according to intuition, the two partial derivatives are always strictly positive (appendix C). There is thus no point to be studied except the points of discontinuity, that is, $\delta = l_{k+1}$ with $k \in \{0, 1, \dots, N-1\}$. Considering the form of the function, it is necessary to find the smallest value of k that maximizes $P_m(\delta = l_{k+1}, \gamma)$ with respect to the following constraints:

for
$$k \in \{0, 1, \dots, N-1\}$$
, $\max_{\gamma} P_m(\delta = l_{k+1}, \gamma)$
s.t.
 $0 \le \gamma \le 1$; $l_0 < l_0^{\max}(k)$ (9)

Highlighting a theoretical solution

In the theoretical plan, the questions are as follows: 1. For a fixed value of M, is there always a regulation of the device that protects the hard core? 2. If yes, what are the corresponding values of K? 3. For every possible value of K, what is the best value of γ ? Is there a value of K and a corresponding value of γ that maximizes the value of P_m ?

$$S_{hc} < \left(1 - M\right) \left(S - \left(\frac{M}{1 - M}\right) \left(\frac{1 - \gamma}{\gamma}\right) \left(\frac{\delta(M - 1)}{M} + S_{hc}^{f}\right)\right)$$

If this condition is not fulfilled, any attempt to take over the target company is bound to fail and any measure to limit the voting rights is therefore pointless. This is why we deliberately place ourselves in the opposite situation.

15. We remind the readers that the situations characterized by either $\delta < l_{N}$ or $\delta \leq l_{0}$ are not treated because they do not present an academic interest.

^{13.} The members of the hard core wish to impose that the hostile investor hold a strictly lower number of shares in *M*.

^{14.} Assuming that the acquisition price offered by the hostile investor is attractive enough, he/she will have no difficulty buying up the shares as long as they are effectively available. The maximum number of shares that the hostile investor can theoretically buy is equal to the total number of shares minus the number of shares in the hands of the hard core shareholders, in other words, $S - S_{hc}$. The takeover will not be attempted unless the defensive threshold is itself below this latter value, in other words, if

^{16.} It is necessary i. to calculate for the function of the local theoretical optima without taking into account constraints, ii. to eliminate all the solutions that do not respect the constraints, iii. to calculate the value of the function for each of the breaking points (superior and lower limits), iv. to confront all these values for those that could be obtained at the end from the third stage, and v. to keep the solution that assures the optimum of the function.

As we mentioned previously,¹⁷ we will concentrate on the particular case where c = 1; this will lead to a simplified theoretical analysis where d = 1 Indeed, in this case, the value of l_0 is independent of P_m and thus of γ , and the analysis is simpler. The cases where c > 1 (with the values of d dependent on those of γ) are treated in the same way but entail a significant amount of discussion without producing a major theoretical contribution.

The value of *M* is fixed, and we look for the values of *k* such that $l_0 < l_0^{\max}(k)$. The calculations made in appendix D allows the assertion that the regulation $\delta = l_{K+1}$ is relevant without being inevitably optimal for

$$M > \frac{l_{k+1}}{S_{hc} + (k+1)l_{k+1} - \sum_{j=1}^{k} l_j}$$
(10)

At this stage, it appears that the hard core's composition has some importance. According to appendix E, the number N of shareholders in the hard core determines the limit that can be reached for M, that is, $M = \frac{1}{N+1}$. In addition, as the fine-tuning of the process is based on the different values of l_i , it is better if no members of the hard core have the same number of shares.

Let us take equation (10), with $\delta = l_{k+1}$, and look for the best possible value of γ We try to maximize P_m while respecting the constraint $l_0 < l_0^{\max}(k)$. If we note $l_0^+ = l_0^- + 1$, equation (F.2) in the appendix F gives us

$$\gamma(l_{k+1}) = \frac{\left(S_{hc} + (k+1)l_{k+1} - \sum_{i=1}^{k} l_i - \frac{l_{k+1}}{M}\right)}{\left(\frac{1-M}{M}\right)l_0^+ + \left(\frac{l_{k+1}(M-1)}{M} + kl_{k+1} - \sum_{i=1}^{k} l_i\right)}$$
(11)

Appendix F shows that this value of γ is always defined and is between 0 and 1 as soon as the inequality $M < \frac{1}{2}$ is verified. To conclude the optimization, it is then enough to determine the couple (δ^*, γ^*) , which verifies

Numerical applications

Here, we calculate the values of δ^* and of γ^* that maximize the market price (P_m) of the share in a numerical configuration chosen arbitrarily and defined by:

The results are obtained thanks to a program developed in VBA (under Excel and with Solver). Eleven values of the price P_0 are considered (decreasing from 100 (P_t) to 150 (bP_t) by 5 points). For every price level, 40 values of *S* are considered (decreasing from 0.50 to 0.11 by 0.01 point). Every value of the parameter *c* represents 440 searches for maximizations with Solver. Appendix G presents an extract of our results for the value $P_0 = 135$ All the results for four various values of *c* are presented in graph 4.

TABLE 1 Numerical applications										
l ₁	100000	l_4	15000	l_7	7500					
l_2	70000	l_5	12500	l ₈	6000					
l ₃	30000	l ₆	10000	l ₉	4000					
S	1000000									
Ν	9									
P_t	100									
а	0.1									
b	1.5									
С	Takes the values 1, 1.2, 1.5, and 2									
М	Decreases from 0.50 to 0.11									

Conclusions and implications for future research

Anti-takeover amendments represent an opportunity for those who wish to protect the national industry against hostile takeover bids. They also facilitate the implementation of strategies for long-term development of firms. However, they put obstacles to the restructuring of company capital and are widely seen as managerial entrenchment mechanisms because they are ostensibly adopted to prevent takeovers that are opposed by management. For these reasons, we examine the limitation measure for voting rights in an environment where takeover defenses have a negative impact on the market that increases with the degree of protection sought. In this study, we analyze, from the perspective of a controlling coalition of incumbent owners whose interests are perfectly aligned, the use of voting restrictions as a mechanism to deter the acquisition of control by a hostile investor. We present a model in which a hard core group of shareholders tries to maintain its control over the capital of a company in the face of a potential takeover bid at a price that may evolve and may directly influence the probability of success.

We show that from the moment we accept the limitation measure for voting rights, the locking device for the capital is rather simple to set up. We build a measure that a group of shareholders could easily compute and implement to prevent takeovers. For that purpose, two parameters are used by the hard core: the threshold from which the voting rights are limited δ and the scale-down coefficient applicable beyond this threshold γ .

First, we model the impacts of the device at three levels: on the actual control of the firm, on the price of the share, and on the attractiveness of the takeover bid. The effects seem contradictory. A regulation with high values of the device risks being ineffective in countering the hostile investor. In contrast, a regulation with low values of the device hampers the hostile investor but may also worry some small shareholders who are then tempted to sell their shares. These sales lower the price of the share and make the offer of the investor more attractive.





The value of c is equal to 1.2



Second, we determine the optimal posture that enables countering the hostile investor's attack and minimizing the counter's negative impact on the market. An optimal regulation (δ^*, γ^*) is highlighted. This solution depends on the level of the investor's offer, on the number of shares held by the members of the hard core, and on the structure of the hard core.

Third, we propose several numerical applications and present various configurations on graphs illustrating the various combinations of δ and γ .

Our study opens new research avenues. Further studies could 1. introduce a second or even a bigger number of hostile investors; 2. give the shareholders of the hard core the ability to exchange shares among themselves; 3. model a conflict among shareholders within the hard core and take into account the potential defection of several members of the coalition; and 4. take into account the quorum, that is, the minimum ratio of the number of voting rights present or represented in the general assembly to the total number of voting rights. Much more work is needed to address the theoretical relevance of these research avenues.







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Appendix

A - Total of voting rights after reduction

$$VR = \sum VR_{i} = VR_{0} + \sum_{i=1}^{K} VR_{i} + \sum_{i=K+1}^{N} VR_{i}$$
$$= \gamma l_{0} + (1 - \gamma)\delta + \sum_{i=1}^{K} \left[\gamma l_{i} + (1 - \gamma)\delta \right] + \sum_{i=K+1}^{N} l_{i}$$
$$= \gamma l_{0} + (1 - \gamma)\delta + \sum_{i=1}^{K} \gamma l_{i} + K(1 - \gamma)\delta + \sum_{i=K+1}^{N} l_{i}$$
$$= \gamma l_{0} + (1 - \gamma)\delta + \left(\sum_{i=1}^{N} \gamma l_{i} - \sum_{i=K+1}^{N} \gamma l_{i} \right) + K(1 - \gamma)\delta + \sum_{i=K+1}^{N} l_{i}$$
$$= \gamma \left(l_{0} + \sum_{i=1}^{N} l_{i} \right) + (1 - \gamma)(\delta + K\delta) + (1 - \gamma) \sum_{i=K+1}^{N} l_{i}$$
$$= \gamma (l_{0} + S_{hc}) + (1 - \gamma)(\delta + S_{hc}^{f})$$

B - Defensive threshold

$$PVR_{0} < M$$

$$\gamma l_{0} + (1 - \gamma)\delta < M\left(\gamma(l_{0} + S_{hc}) + (1 - \gamma)(\delta + S_{hc}^{f})\right)$$

$$\gamma l_{0}(1 - M) < M\gamma S_{hc} + (1 - \gamma)(M\delta - \delta + MS_{hc}^{f})$$

$$l_{0} < \frac{M}{1 - M}S_{hc} + \frac{(1 - \gamma)}{\gamma(1 - M)}(\delta(M - 1) + MS_{hc}^{f})$$
We note $l_{0}^{\max} = \left(\frac{M}{1 - M}\right)\left(S_{hc} + \left(\frac{1 - \gamma}{\gamma}\right)\left(\frac{\delta(M - 1)}{M} + S_{hc}^{f}\right)\right)$

We obtain $l_0 < l_0^{\max}$. For $M = \frac{1}{2}$ and $\gamma = 1$ we find again the inequality $l_0 < S_{hc}$ according to the hypotheses of the model.

C - The gradient of the price function

$$P_{m} = \frac{P_{t}}{Ln(P_{t})} \times Ln \left[aP_{t} \frac{\delta}{S} + (1-a)P_{t} \left(1 - \left(1 - \frac{\delta}{S} \right)(1-\gamma) \right) \right]$$

We look for the values of cancellation of the function's gradient to obtain the local extremums.

$$\frac{\partial P_m}{\partial \delta} = \frac{P_t}{Ln(P_t)} \times Ln \left[\frac{(P_t / S)(a + (1 - a)(1 - \gamma))}{aP_t(\delta / S) + (1 - a)P_t(1 - (1 - (\delta / S))(1 - \gamma))} \right] > 0$$
$$\frac{\partial P_m}{\partial \gamma} = \frac{P_t}{Ln(P_t)} \times Ln \left[\frac{(1 - a)P_t(1 - (\delta / S))}{aP_t(\delta / S) + (1 - a)P_t(1 - (1 - (\delta / S))(1 - \gamma))} \right] > 0$$

The two partial derivatives are always strictly positive. There is no value of cancellation of the gradient.

D – The value of δ which can allow to reach the optimum

The optimization is possible only for values of δ which correspond to the borders of the intervals of definition of $l_0^{\max}(k)$. Thus we have to study successively the cases $\delta = l_{k+1}$ for $k \in \{0, ..., N-1\}$.

For k = 0 the value of δ which can allow to reach the optimum is $\delta = l_1$. From then the constraint on l_0 becomes $l_0 \le l_0^{\max}(0)$:

$$\begin{split} l_0 &< \left(\frac{M}{1-M}\right) \left(S_{hc} + \left(\frac{1-\gamma}{\gamma}\right) \left(\frac{l_1(M-1)}{M} + S_{hc}\right)\right) \\ &\left(\frac{1-M}{M}\right) l_0 - S_{hc} < \left(\frac{1}{\gamma} - 1\right) \left(\frac{l_1(M-1)}{M} + S_{hc}\right) \\ &\frac{(l_0 - l_1)(1-M)}{M} < \frac{1}{\gamma} \left(\frac{l_1(M-1)}{M} + S_{hc}\right) \end{split}$$

For $l_0 \le l_1$, the model does not present interest; consequently we consider the case $l_0 > l_1$. Besides $0 < M \le \frac{1}{2}$ then $(l_0 - l_1)(1 - M) / M > 0$, we obtain

$$\gamma < \frac{\frac{l_1(M-1)}{M} + S_{hc}}{\frac{(l_0 - l_1)(1 - M)}{M}}$$

Given that $\gamma \ge 0$ and as the denominator is positive, this last inequality is the true as long as $(l_1(M-1)/M) + S_{hc} > 0$, so that At the level of this value and down, $M > l_1 / (S_{hc} + l_1)$. cannot be any more an optimum and we have to study then successively the cases $\delta = l_{k+1}$ for $k \in \{1, ..., N-1\}$.

For k = 1, the value of δ which can allow to reach the optimum is $\delta = l_{2}$. From then the constraint on l_{0} becomes $l_{0} \leq l_{0}^{\max}(1)$, then :

$$l_{0} < \left(\frac{M}{1-M}\right) \left(S_{hc} + \left(\frac{1-\gamma}{\gamma}\right) \left(\frac{l_{2}(M-1)}{M} + S_{hc} + l_{2} - l_{1}\right)\right)$$
$$\left(\frac{1-M}{M}\right) l_{0} - S_{hc} < \left(\frac{1}{\gamma} - 1\right) \left(\frac{l_{2}(M-1)}{M} + S_{hc} + l_{2} - l_{1}\right)$$
$$\frac{(l_{0} - l_{2})(1-M)}{M} + l_{2} - l_{1} < \frac{1}{\gamma} \left(\frac{l_{2}(M-1)}{M} + S_{hc} + l_{2} - l_{1}\right)$$

we note that

$$\frac{(l_0 - l_2)(1 - M)}{M} + l_2 - l_1 = \frac{(l_0 - l_1 + l_1 - l_2)(1 - M)}{M} + l_2 - l_1$$

then

$$\frac{(l_0 - l_2)(1 - M)}{M} + l_2 - l_1 = (l_0 - l_1)\frac{(1 - M)}{M} + (l_1 - l_2)\frac{(1 - 2M)}{M}$$

as
$$M \le \frac{l_1}{(S_{hc} + l_1)} < \frac{l_1}{(l_1 + l_1)} = \frac{1}{2}$$
 and $l_0 > l_1 > l_2$ and $0 < M$

then
$$\frac{(l_0 - l_2)(1 - M)}{M} + l_2 - l_1 > 0 \text{ so that}$$
$$\gamma < \frac{\frac{l_2(M - 1)}{M} + S_{hc} + l_2 - l_1}{\frac{(l_0 - l_2)(1 - M)}{M} + l_2 - l_1}$$

Given that $\gamma \ge 0$ and as the denominator is positive, this last inequality is the true as long as

 $\begin{aligned} &(l_2(M-1)/M)+S_{hc}+l_2-l_1>0\text{ , then }M>l_2/(S_{hc}+2l_2-l_1).\\ &\text{At the level of this value and down, }\delta=l_2 \text{ cannot be any more}\\ &\text{an optimum and we have to study then successively the cases}\\ &\delta=l_{k+1} \text{ for }k\in\left\{2,...,N-1\right\}. \end{aligned}$

For k = 1, the value of δ which can allow to reach the optimum is $\delta = l_3$. From then the constraint on l_0 becomes $l_0 \leq l_0^{\max}(2)$, then :

we note that

$$\frac{(l_0 - l_3)(1 - M)}{M} + 2l_3 - l_2 - l_1 = \frac{(l_0 - l_1 + l_1 - l_2 + l_2 - l_3)(1 - M)}{M} + 2l_3 - 2l_2 + l_2 - l_1$$

then

$$\frac{(l_0 - l_3)(1 - M)}{M} + 2l_3 - l_2 - l_1 = (l_0 - l_1)\frac{(1 - M)}{M} + (l_1 - l_2)\frac{(1 - 2M)}{M} + (l_2 - l_3)\frac{(1 - 3M)}{M}$$

as
$$M \le \frac{l_2}{(S_{hc} + 2l_2 + l_1)} < \frac{l_2}{(l_2 + 2l_2)} = \frac{1}{3}$$
 and $l_0 > l_1 > l_2 > l_3$ and

then
$$\frac{(l_0 - l_3)(1 - M)}{M} + 2l_3 - l_2 - l_1 > 0 \text{ so that}$$
$$\gamma < \frac{\frac{l_3(M - 1)}{M} + S_{hc} + 2l_3 - l_2 - l_1}{\frac{(l_0 - l_3)(1 - M)}{M} + 2l_3 - l_2 - l_1}$$

Given that $\gamma \ge 0$ and as the denominator is positive, this last inequality is the true as long as

 $(l_3(M-1)/M) + S_{hc} + 2l_3 - l_2 - l_1 > 0$, then $M > l_3 / (S_{hc} + 3l_3 - l_2 - l_1)$. According to a recurrence, the regulation $\delta = l_{k+1}$ can be an optimum only if:

$$M > \frac{l_{k+1}}{S_{hc} + (k+1)l_{k+1} - \sum_{i=1}^{k} l_i}$$
(D.1)

E - Asymptotic behavior of the device

From the inequality (D.1) of the appendix D, it seems that the smallest value for *M* is:

F – Research of the optimal value of γ

Let us place now in the situation (D.1) for which $\delta = l_{k+1}$ and let us look for the best value of γ , the one who maximizes P_m while verifying the constraint $l_0 < l_0^{\max}(k)$. According appendix B, we know that $(\partial P_m / \partial \gamma) > 0$. We look for the biggest value of γ which verifies $l_0 < l_0^{\max}(k)$. On the interval of study, $l_0^{\max}(k)$ is equal to :

$$l_{0}^{\max}(k) = \left(\frac{M}{1-M}\right) \left(S_{hc} + \left(\frac{1}{\gamma} - 1\right) \left(\frac{l_{k+1}(M-1)}{M} + S_{hc} + kl_{k+1} - \sum_{i=1}^{k} l_{i}\right)\right)$$

so that $\frac{\partial l_{0}^{\max}(k)}{\partial \gamma} = \frac{-1}{\gamma^{2}} \left(\frac{M}{1-M}\right) \left(\frac{l_{k+1}(M-1)}{M} + S_{hc} + kl_{k+1} - \sum_{i=1}^{k} l_{i}\right)$

As $0 < M < \frac{1}{2}$, the derivative has an opposite sign compared with: $l_{k+1}(M-1) / M + S_{hc} + kl_{k+1} - \sum_{i=1}^{k} l_i = S_{hc} + (k+1)l_{k+1} - \sum_{i=1}^{k} l_i - (l_{k+1} / M)$ But according to (D.1), on the interval of study, we know that $M > l_{k+1} / (S_{hc} + (k+1)l_{k+1} - \sum_{i=1}^{k} l_i)$, consequently:

$$S_{hc} + (k+1)l_{k+1} - \sum_{i=1}^{k} l_i - \frac{l_{k+1}}{M} > 0$$
(F.1)

and finally
$$\frac{\partial l_0^{\max}(k)}{\partial \gamma} < 0$$

As we wish to hold the biggest value of γ possible, it is necessary to retain the least binding value of $l_0^{\max}(k)$ which respects the constraint $l_0 < l_0^{\max}(k)$, that is $l_0^{\max}(k) = l_0 + 1$. Afterward we note $l_0 + 1 = l_0^+$ and we must solve the equation:

$$I_{0}^{\max}(k) = \left(\frac{M}{1-M}\right) \left(S_{hc} + \left(\frac{1}{\gamma} - 1\right) \left(\frac{l_{k+1}(M-1)}{M} + S_{hc} + kl_{k+1} - \sum_{i=1}^{k} l_{i}\right)\right) = l_{0}^{k}$$

$$\frac{1}{\gamma} \left(\frac{l_{k+1}(M-1)}{M} + S_{hc} + kl_{k+1} - \sum_{i=1}^{k} l_{i}\right) = \left(\frac{1-M}{M}\right) l_{0}^{k} - S_{hc} + \left(\frac{l_{k+1}(M-1)}{M} + S_{hc} + kl_{k+1} - \sum_{i=1}^{k} l_{i}\right)$$

$$\frac{1}{\gamma} \left(S_{hc} + (k+1)l_{k+1} - \sum_{i=1}^{k} l_{i} - \frac{l_{k+1}}{M}\right) = \left(\frac{1-M}{M}\right) l_{0}^{k} + \left(\frac{l_{k+1}(M-1)}{M} + kl_{k+1} - \sum_{i=1}^{k} l_{i}\right)$$
We study now: $\left(\frac{1-M}{M}\right) l_{0}^{k} + \left(\frac{l_{k+1}(M-1)}{M} + kl_{k+1} - \sum_{i=1}^{k} l_{i}\right)$

For k = 0, we obtain:

$$\left(\frac{1-M}{M}\right)l_0^+ + \left(\frac{l_1(M-1)}{M}\right) = \left(\frac{1-M}{M}\right)(l_0^+ - l_1) > 0 \text{ because}$$

 $M \leq \frac{1}{2} < 1$ on the interval of study

For
$$k = 1$$
, we obtain:
 $\left(\frac{1-M}{M}\right)l_0^+ + \left(\frac{l_2(M-1)}{M} + l_2 - l_1\right) = \left(\frac{1-M}{M}\right)l_0^+ - \left(\frac{1-M}{M}\right)l_1 + \left(\frac{1-M}{M}\right)l_1 - \left(\frac{1-M}{M}\right)l_2 + l_2 - l_1$
then $\left(\frac{1-M}{M}\right)(l_0^+ - l_1) + \left(\frac{1-M}{M}\right)l_1 - \left(\frac{1-2M}{M}\right)(l_1 - l_2) > 0$
because $M \le \frac{l_1}{S_{hc}} + l_1 < \frac{1}{2}$ on the interval of study.

For k = 2, we obtain:

$$\left(\frac{1-M}{M}\right) l_0^+ + \left(\frac{l_3(M-1)}{M} + 2l_3 - l_1 - l_2\right) \text{ then}$$

$$\left(\frac{1-M}{M}\right) l_0^+ - \left(\frac{1-M}{M}\right) l_1 + \left(\frac{1-M}{M}\right) l_1 - \left(\frac{1-2M}{M}\right) l_2 + \left(\frac{1-2M}{M}\right) l_2 + \frac{l_3(M-1)}{M} + 2l_3 + l_2 - l_1$$

$$\text{ then } \left(\frac{1-M}{M}\right) (l_0^+ - l_1) + \left(\frac{1-2M}{M}\right) (l_1 - l_2) + \left(\frac{1-3M}{M}\right) (l_2 - l_3) > 0$$

$$\text{ because } M \le \frac{l_2}{S_{hc} + 2l_2 - l_1} < \frac{1}{3} \text{ on the interval of study.}$$

Let us write a relation of recurrence (with M < 1/(k+1) on the interval of study):

$$\left(\frac{1-M}{M}\right)l_0^+ + \left(\frac{l_{k+1}(M-1)}{M} + kl_{k+1} - \sum_{i=1}^k l_i\right) > 0$$

For $\delta = l_{k+1}$, thus we can always define an optimal value of γ by the equality:

$$\gamma(l_{k+1}) = \frac{\left(S_{hc} + (k+1)l_{k+1} - \sum_{i=1}^{k} l_i - \frac{l_{k+1}}{M}\right)}{\left(\frac{1-M}{M}\right)l_0^{+} + \left(\frac{l_{k+1}(M-1)}{M} + kl_{k+1} - \sum_{i=1}^{k} l_i\right)}$$
(F.2)

This value is always positive – the numerator is also positive according to (F.1) $_{\overline{k}}$ and lower than 1. Indeed, if we write $D = (k+1)I_{k+1} - \sum_{i=1}^{k} l_i - \frac{l_{k+1}}{M}$, then $\gamma(l_{k+1})$ becomes:

$$\gamma(l_{k+1}) = \frac{S_{hc} + D}{\left(\frac{1 - M}{M}\right)l_0^+ + D}$$

Yet, by construction $S_{_{hc}} < (1-M)l_0^+ / M$ since $M < \frac{1}{2}$. To conclude the optimization, it is then enough to determine the couple (δ^*, γ^*) which verifies:

for
$$k \in \{0, 1, ..., N-1\}$$
, $\max P_m(\delta = l_{k+1}, \gamma = \gamma(l_{k+1}))$ (F.3)

	G – The value of δ^* and γ^* for c = 1; c = 1.2; c = 1.5 then c = 2 for P ₀ = 135												
			c = 1			c = 1.2			c = 1.5			c = 2	
P0	S	Delta	Gamma	Pm	Delta	Gamma	Pm	Delta	Gamma	Pm	Delta	Gamma	Pm
135	0.50	100000	0.368	79.987	100000	0.390	80.932	100000	0.431	82.626	100000	0.529	86.144
135	0.49	100000	0.344	78.912	100000	0.364	79.811	100000	0.402	81.434	100000	0.490	84.811
135	0.48	100000	0.321	77.825	100000	0.339	78.679	100000	0.373	80.229	100000	0.453	83.465
135	0.47	100000	0.299	76.726	100000	0.315	77.534	100000	0.346	79.010	100000	0.418	82.104
135	0.46	100000	0.278	75.612	100000	0.292	76.374	100000	0.320	77.776	100000	0.385	80.724
135	0.45	100000	0.258	74.482	100000	0.271	75.197	100000	0.295	76.524	100000	0.353	79.326
135	0.44	100000	0.238	73.333	100000	0.249	74.001	100000	0.272	75.252	100000	0.323	77.905
135	0.43	100000	0.219	72.164	100000	0.229	72.785	100000	0.249	73.960	100000	0.294	76.461
135	0.42	100000	0.201	70.972	100000	0.210	71.546	100000	0.227	72.643	100000	0.267	74.991
135	0.41	100000	0.183	69.754	100000	0.191	70.282	100000	0.206	71.300	100000	0.241	73.492
135	0.40	100000	0.166	68.509	100000	0.173	68.989	100000	0.186	69.928	100000	0.216	71.961
135	0.39	100000	0.150	67.232	100000	0.155	67.665	100000	0.166	68.523	100000	0.192	70.395
135	0.38	100000	0.134	65.921	100000	0.138	66.306	100000	0.148	67.084	100000	0.170	68.791
135	0.37	100000	0.118	64.572	100000	0.122	64.909	100000	0.130	65.605	100000	0.148	67.146
135	0.36	100000	0.103	63.179	100000	0.106	63.470	100000	0.113	64.083	70000	0.161	65.530
135	0.35	100000	0.089	61.738	100000	0.091	61.983	100000	0.096	62.513	70000	0.143	63.947
135	0.34	100000	0.074	60.243	100000	0.076	60.443	100000	0.080	60.890	70000	0.127	62.326
135	0.33	100000	0.061	58.687	100000	0.062	58.844	70000	0.100	59.352	70000	0.112	60.664
135	0.32	70000	0.082	57.099	70000	0.084	57.298	70000	0.087	57.791	70000	0.097	58.955
135	0.31	70000	0.071	55.597	70000	0.072	55.758	70000	0.075	56.180	70000	0.083	57.196
135	0.30	70000	0.060	54.037	70000	0.061	54.160	70000	0.063	54.512	70000	0.069	55.380
135	0.29	70000	0.050	52.409	70000	0.050	52.498	70000	0.052	52.782	70000	0.057	53.502
135	0.28	70000	0.040	50.706	70000	0.040	50.763	70000	0.041	50.981	30000	0.064	46.677
135	0.27	70000	0.030	48.914	70000	0.030	48.945	70000	0.031	49.101	30000	0.056	44.931
135	0.26	70000	0.021	47.022	70000	0.021	47.033	30000	0.047	42.478			
135	0.25	70000	0.011	45.013	30000	0.040	40.624	30000	0.040	40.738			
135	0.24	30000	0.035	38.972	30000	0.034	38.865	30000	0.034	38.932			
135	0.23	30000	0.029	37.151	30000	0.029	37.027	15000	0.031	31.596			
135	0.22	30000	0.024	35.233	30000	0.023	35.098						
135	0.21	30000	0.018	33.202	30000	0.018	33.064						
135	0.20	30000	0.013	31.038	15000	0.021	26.206						
135	0.19	30000	0.009	28.714	15000	0.017	24.188						
135	0.18	30000	0.004	26.198	15000	0.014	22.053						
135	0.17	15000	0.012	20.101									
135	0.16	15000	0.008	17.631									
135	0.15	15000	0.006	14.940									
135	0.14	15000	0.003	11.973									
135	0.13	12500	0.002	7.676									
135	0.12	10000	0.001	2.146									
135	0.11												