Review of Economic Analysis

REA Review of Economic Analysis

# On 'Rusting' Money: Silvio Gesell's Schwundgeld Reconsidered. Part II: The Long Run

Günther Rehme

Volume 16, Number 2, 2024

URI: https://id.erudit.org/iderudit/1113986ar
DOI: https://doi.org/10.15353/rea.v16i2.4945

See table of contents

Publisher(s)

International Centre for Economic Analysis

ISSN

1973-3909 (digital)

Explore this journal

Cite this article

Rehme, G. (2024). On 'Rusting' Money: Silvio Gesell's Schwundgeld Reconsidered. Part II: The Long Run. *Review of Economic Analysis*, *16*(2), 133–174. https://doi.org/10.15353/rea.v16i2.4945

Article abstract

Silvio Gesell argued that 'rusting' money is economically and socially beneficial; that claim has often been contended. In Part II of the paper, I concentrate on the long-run implications of his ideas. I show that introducing money depreciation in isolation may be economically non-beneficial in typical long-run equilibrium. But money depreciation, when coupled with expansionary monetary policy, is a necessary condition for a positive Mundell-Tobin effect on long-run real variables and so creates wealth in the model. It is found that this also holds in the transition to the long-run equilibrium. Hence, the spirit of Gesell's hypotheses can be verified for a plausible, long-run environment as well, and may, thus, be relevant for long-run economic policy problems.

Review of Economic Analysis 16 (2024) 133-173 1973-3909/2024133

# On 'Rusting' Money: Silvio Gesell's *Schwundgeld* Reconsidered.

## Part II: The Long Run

**GÜNTHER REHME**[†]
*Technische Universität Darmstadt* *

Silvio Gesell argued that 'rusting' money is economically and socially beneficial; that claim has often been contended. In Part II of the paper, I concentrate on the long-run implications of his ideas. I show that introducing money depreciation in isolation may be economically non-beneficial in a typical long-run equilibrium. But money depreciation, when coupled with expansionary monetary policy, is a necessary condition for a positive Mundell-Tobin effect on long-run real variables and so creates wealth in the model. It is found that this also holds in the transition to the long-run equilibrium. Hence, the spirit of Gesell's hypotheses can be verified for a plausible, long-run environment as well, and may, thus, be relevant for long-run economic policy problems.

*Keywords*: Economic Performance, Depreciating Money, Zero Lower Bound, Demonetization, Love of Wealth

*JEL classification*: E1, E5, O4

---

[†] Professor Rehme passed away last year. The paper is published as it was originally submitted, with the exception of the quotes in part D, which are the same as the first part of the work, pages 91-131 in this issue.

* I am indebted to Ingo Barens and Thomas Fischer for valuable help and insightful comments. I have also benefitted from discussions with Parantab Basu, Volker Caspari, Christiane Clemens, Alex Cukierman, Soumya Datta, Hartmut Egger, Sabine Eschenhof-Kammer, Christian Gelleri, Rafael Gerke, Chetan Ghate, Charles Goodhart, Marcus Miller, Michael Neugart, Uwe Sunde, Werner Onken, and from feedback at Bayreuth, LMU Munich, the 5th International Conference on South Asian Economic Development (SAED), South Asian University (SAU), New Delhi, the 5th HenU/INFER Workshop on Applied Macroeconomics, Kaifeng, Henan, the 50th Anniversary "Money, Macro and Finance" (MMF) conference at the London School of Economics (LSE), London, the 10th RCEA "MoneyMacro-Finance" Conference in Waterloo, Ontario, the 15th Annual Conference on Economic Growth and Development, New Delhi, in 2019, and the 65th Münden Talks "Proudhon, Gesell, Keynes and Negative Interest Rates", Wuppertal, in 2021. The usual disclaimer shields them all.

## 1 Introduction

"Money is the football of economic life."

Silvio Gesell (1920)
*The Natural Economic Order.*

In his main piece of work, "The Natural Economic Order" Silvio Gesell developed his idea of Schwundgeld (demurrage) and its consequences on economic performance. In part I of the paper it is shown that Gesell's claim can be justified in a short-run IS-LM-AS-AD environment. In Part II I now analyze whether his key conjectures can be justified in a parsimonious, modern theoretical framework for the long run.

Gesell (1920), p. 78, acknowledges that money is "the football of economic life", but to him placing money and commodities on equal 'physical' footing as commodities is necessary and requires that money depreciate so that it performs its prime task, namely that of being the medium of exchange. For him, the face value of (paper) fiat money should be irredeemable and depreciate at a certain percentage over a particular period of time. In order to regain the previous face value of the money (note) used, people would have to buy stamps to make up for the depreciation the monetary authority would decree for the money note. As pointed out in Part I Gesell formulated four hypotheses about such a monetary arrangement. [1]

**Gesell Conjecture 1 (GC1)** *The introduction of, and, when present, an increase in, the money depreciation rate leads to a higher velocity of money in circulation.*

**Gesell Conjecture 2 (GC2)** *Money depreciation coupled with expansionary monetary policy stimulates aggregate demand and through that output and employment.*

**Gesell Conjecture 3 (GC3)** *A money depreciation rate is welfare enhancing.*

**Gesell Conjecture 4 (GC4)** *A money depreciation rate benefits workers relatively more than capital owners.*

The present paper complements research that investigates whether the Gesell hypotheses can be replicated in modern standard model frameworks for the long run. One finds that the results of previous research are mixed. [2]

For example, Rösl (2006) finds that only the first hypothesis can be derived from Sidrauski (1967), that is, in a money-in-the-utility set-up. He concludes that Gesell neglected an analysis

[1] More verbal justifications for Gesell's claims and his ideas can be found in the working paper version of this paper; see Rehme (2018), especially appendix F, and at the end of this paper.

[2] Gesell's ideas have been important in recent discussions about overcoming the problems after the Great Recession. For good surveys on the relevance of Gesell's ideas see, for example, Darity (1995), Ilgmann and Menner (2011), and Svensson and Westermark (2016).

of the long run and any possible effects on capital accumulation so the other three hypotheses turn out to be non-valid in his model.

In turn, Menner (2011), for example, uses an elaborate and involved New Monetarist DSGE model to find that "inflation and 'Gesell taxes' maximize steady-state capital stock, output, consumption, investment and welfare at moderate levels. ... In a recession scenario, a Gesell tax speeds up the recovery in a similar way as a large fiscal stimulus but avoids 'crowding out' of private consumption and investment." Thus, he finds support for the Gesell hypotheses at moderate levels in his business cycle model of the third-generation monetary search models.

The present paper uses an alternative micro-founded and simple dynamic general equilibrium model to analyze whether the depreciation of money is socially beneficial. For that, we abstract from fiscal policy, as Gesell did not consider the interaction of fiscal and monetary policy in detail. Following him we assume that the state issues a homogenous money and by legal coercion that money is legal tender. That is explained in some detail in Part I of this contribution.

In the present paper, the basic Sidrauski framework is coupled with the additional motive of an agent to derive utility from (real) wealth. [3] People are taken to be rational and are not fooled by money illusion. Thus, the agents only consider real, physical capital as wealth. In that way, I relate to this as a 'love of wealth' as in Rehme (2011).

In Part II I use a standard Ramsey-Cass-Koopmans framework where markets are assumed to clear at each point in time, and demand equals supply. Importantly, and as is standard in the literature, the marginal productivity theory of distribution is (now) assumed to hold in this model framework. It turns out that this yields interesting insights about Gesell's advocated monetary system where fiat (paper) money is irredeemable and, thus, directly related to a basket of real goods in an economy. [4] These insights refer, in particular, to the idea of money depreciation and its consequences for the steady state of an economy and its transitional dynamics. In such an optimal growth framework these results then emerge.

[3] This has been done, for example, by Weber (1930) and Pigou (1941) who argue that individuals derive utility from the mere possession of wealth and not simply its expenditure. Later Kurz (1968) provided a thorough analysis of an optimal growth model where wealth features in utility. Furthermore, Zou (1994), Bakshi and Chen (1996) and Carroll (2000) relate to Max Weber and argue that the dependence of utility on wealth captures the "spirit of capitalism" in competitive market economies. More generally, it captures the 'love of wealth' in more general set-ups, including competitive market economies, as argued in Rehme (2017).

[4] Irredeemability implies that you cannot exchange a banknote back into another banknote or any collateral that might possibly back the face or any other (real) value of the banknote. For instance, in the Euro and the Fed system you can in principle redeem your banknote, but only to get another banknote with an equally denoted face value. This is not possible under the irredeemability of the banknote and plays a role when there is a depreciation of the face or other value of the banknote. On the issue of irredeemability and fiat money see, for example, Buiter (2003).

In the steady state, inflation depends on the sum of the money growth and depreciation rate. It turns out that the model dichotomizes into a monetary and real sector if there is no money depreciation. If the latter is present, the model features non-superneutrality. Thus, in the model money depreciation is a necessary condition for particular forms of a *Mundell-Tobin* effect. That effect is present if inflation leads people to hold less money and more real capital, implying a lower real interest rate.

More precisely, it is found that the introduction of or an increase in money depreciation in isolation reduces the steady state capital stock (wealth), consumption, income and welfare. It also implies a higher return to capital, but a lower steady-state wage rate. Thus, more money depreciation seems to destroy wealth and implies lower wages. The only hypothesis that is validated is that higher money depreciation implies a higher velocity of money, [GC1].

Some authors have stopped here to argue that money depreciation is generally a bad idea, because it just destroys long-run wealth, instead of fostering it. However, in light of the quotes above, that view does not do justice to Gesell's thinking. He was not arguing solely about money depreciation. Of course, he knew that the monetary authority was also issuing new and withdrawing old money.

Here it turns out that, for a given positive money depreciation rate, an increase in the money growth rate produces a *Mundell-Tobin* effect. Thus, higher money growth increases steady-state inflation, but also the steady-state capital stock, output, and consumption. It implies a higher long-run wage rate and a lower return to capital. The consequences for the holdings of real money balances and so for total welfare are not unambiguously clear. But the velocity of money increases. However, the partial welfare channels through consumption and wealth work clearly in a positive direction.

Hence, the conjectures GC1 and GC2 can be validated for the long run. But given the necessary nature of money depreciation for these results one may argue that GC3 and GC4 are also not too far off their marks. In terms of the economic effects the conjectures ultimately wish to capture they are not wrong because of the possibility of a positive *Mundell-Tobin* effect which would indeed support GC3 and GC4.

The analysis of the transitional dynamics reveals that the speed of convergence increases if money depreciation increases, and decreases if the money growth rate is raised. That complements Fischer (1979) who finds that more money growth speeds up convergence when utility is non-logarithmic and the steady state features asymptotic superneutrality. Here the steady state generally features non-superneutrality, utility is logarithmic, and convergence is slower when the money growth rate increases.

A simulation exercise based on some standard calibration values reveals that the response of the key variables to permanent changes in the monetary policy variables is the same in the transition as in the steady-state. That also holds for the jump variables, namely, initial money holdings and consumption.

Furthermore, for temporary changes in the policy variables, one obtains the temporary responses that, again, qualitatively equal those for the steady state.

Summarizing these findings yields that the present model framework is indeed capable of verifying most of Gesell's claims, also in the long run. In Part II and, thus, for a long-run equilibrium two claims of Gesell's follow directly, and the other two indirectly, because money depreciation is a necessary condition for a positive *Mundell-Tobin* effect. This may justify why Gesell's ideas may have significance for a description of long-run macroeconomic phenomena and realistically relate to the current economic situation in many countries.

The paper is organized as follows. Section 2 presents the model and its set up. Section 3 derives and analyzes the long-run equilibrium and section 4 the transitional dynamics. Section 5 concludes.

## 2 The Model

The set up of the model is explained in detail in Part I of the paper. For the purposes of Part II, I only restate the main ingredients of the model. I use a continuous time framework and for all variables that are continuous functions of time the subscript $t$ is used to denote their dependence on time. Thus, $h_t \equiv h(t)$ for some variable $h$ depending on time. Furthermore, the change of a variable $h$ over time, i.e. $\frac{dh_t}{dt}$, is denoted by $\dot{h}_t$.

The economy has many, price-taking households. The aggregate resource constraint of the households is

$$C_t + \dot{K}_t + \frac{\dot{M}_t}{P_t} + \sigma \cdot \frac{M_t}{P_t} = w_t N_t + r_t K_t + X_t \tag{1}$$

where $C_t$ and $K_t$ denote aggregate real consumption and the aggregate real capital stock, respectively. $M_t$ represents the aggregate nominal money holdings and $P_t$ is the price level. $N_t$ denotes population and $w_t$ is the real wage rate. $r_t$ denotes the real rate of return on capital, net of depreciation of physical capital $K_t$. The lump-sum (real) transfers of the government are denoted $X_t$.

Thus, the right-hand side of the budget constraint captures aggregate income, consisting of total wage $(w_t N_t)$ and capital income $(r_t K_t)$ as well as government transfers $(X_t)$ and the left-hand side, captures aggregate spending. Thus, income is spent on consumption $(C_t)$, investment in new capital $(\dot{K}_t)$ and acquisitions of new, real money holdings $\left(\frac{\dot{M}_t}{P_t}\right)$.

The aggregate budget constraint in equation (1) corresponds to the conventional money-in-the-utility-function model. The novel feature here is the term $\sigma \cdot \frac{M_t}{P_t}$. It captures the Gesell tax and so the idea of *"rusting money"*. That can be interpreted as a depreciation on the circulating real money holdings of the households and is tantamount to a tax on them.

Now consider a representative agent economy, and define per capita consumption $c_t$, real money balances $m_t$, as well as the per capita capital stock $k_t$ and transfers $x_t$ as follows

$$c_t \equiv \frac{C_t}{N_t},\ m_t \equiv \frac{M_t}{P_t N_t},\ k_t \equiv \frac{K_t}{N_t},\ \text{and } x_t \equiv \frac{X_t}{N_t}$$

One verifies that the budget constraint of the representative household is then given by

$$c_t + \dot{k}_t + n_t k_t + \dot{m}_t + \pi_t m_t + n_t m_t + \sigma m_t = w_t + r_t k_t + x_t.$$

Again, the right-hand side corresponds to the household's income and the left-hand side captures the household's expenditure. Notice $\sigma m_t$ is the outlay for the household. The longer the household holds real money balances $m_t$, the more is foregone (a form of expenditure) in terms of real income.[5]

For simplicity let $a_t \equiv k_t + m_t$. Thus, the household has real resources in the form of physical capital and real money balances. Then $\dot{a}_t = \dot{k}_t + \dot{m}_t$. After collecting terms and rearrangement one then obtains

$$\dot{a}_t = [(r_t - n_t)a_t + w_t + x_t] - [c_t + (r_t + \pi_t + \sigma)m_t]. \tag{2}$$

Thus, the change in real per capita resources $\dot{a}_t$ depends on the household's income from capital and real money balances $(r_t - n_t)a_t$, labor income $w_t$ and transfers $x_t$. Consumption then consists of the consumption of goods $c_t$ and the expenses for using money services. The latter depends on the user cost of money $(r_t + \pi_t + \sigma)m_t$. Here we employ the Fisher relation that nominal interest rates $i_t$ equal the real interest rate $r_t$ plus the inflation rate $\pi_t$. The user cost of holding money, thus, depends on the nominal interest rate $i_t$ and the depreciation of money $\sigma$. To simplify the analysis assume a stationary population $n_t = 0$ and set its size to $N_t = 1$ for all $t$.

As an important departing point from a standard Sidrauski model the representative household also "loves wealth". The household is not fooled by money illusion and only physical capital is considered to be "wealth" that directly bears on welfare.

However, the household also values real money balances as they facilitate exchange and transactions. Thus, (real) money balances are also taken to bear on welfare as in Sidrauski (1967). Although both money and capital feature directly in utility, they do so for different reasons. Money is valued because it facilitates exchange, whereas physical capital is valued as an expression of wealth.

[5] To capture the Gesell tax in this way see, for example, Rösl (2006), and the explanations in Part I.

The household's problem is then taken to be to maximize the functional

$$W = \int_0^\infty \varphi(c_t, m_t, k_t) e^{-\rho t} dt \tag{3}$$

where $\varphi(c_t, m_t, k_t)$ is period utility depending on consumption, real money balances and physical capital. Welfare is discounted at the (positive) rate of time preference $\rho$, capturing how patient households are, and the convergence of the utility function.

In order to derive clear predictions that also allow for an analysis of transitional dynamics, and building on previous own work, cf. Rehme (2011), we now make the following assumptions about the period utility function $\varphi(c_t, m_t, k_t)$.

1. $\varphi(c_t, m_t, k_t)$ is taken to be separable in $c_t, m_t$ and $k_t$. In particular, assume that of the project.

$$\partial^2 \varphi(\cdot) / \partial i \, \partial j = 0 \text{ for all } i, j = c_t, m_t, k_t \text{ and } i \neq j.$$

2. $\varphi(c_t, m_t, k_t)$ is increasing and concave in each (own) argument, that is,

$$\partial \varphi(\cdot) / \partial i > 0 \text{ and } \partial^2 \varphi(\cdot) / \partial i^2 < 0 \text{ for all } i = c_t, m_t, k_t.$$

3. $\varphi(c_t, m_t, k_t)$ satisfies the Inada conditions for each (own) argument, that is,

$$\lim_{i \to 0} \varphi(\cdot) / \partial i \to \infty \text{ and } \lim_{i \to \infty} \varphi(\cdot) / \partial i \to 0 \text{ where } i = c_t, m_t, k_t.$$

A simple and convenient period utility function that satisfies all these requirements is the logarithmic one. So we invoke

**Assumption 1** Period utility $\varphi(c_t, m_t, k_t)$ is separable and logarithmic in each argument and given by

$$\varphi(c_t, m_t, k_t) = \ln\, c_t + \delta \ln\, m_t + \beta \ln\, k_t \text{ where } \delta, \beta > 0 \tag{4}$$

The parameter $\delta$ measures how people value the transaction services real money balanced render, and $\beta$ captures "love of wealth". The assumption that $\delta$ and $\beta$ are positive means that the model is structurally different from the more conventional setups of "money-in-the-utility-function"-models without "love of wealth". [6]

[6] From the logarithmic utility set-up it is immediate that relative wealth, for instance, the logarithm of the ratio of individual to total (aggregate) wealth would be separable in the two concepts. If the representative individual takes total wealth as given, then both approaches, that is, working with relative

Let $[h_t]_{t=0}^{+\infty}$ denote the continuous time path of variable $h_t$ and use the following definitions: $k_t \equiv (1-z_t)a_t$ and $m_t \equiv z_t a_t$ where $a_t$ is an indicator of the total real resources of the household, and $z_t$ denotes the share of the real resources held in terms of real money balances. These definitions serve to facilitate the analysis, and i.a. imply

$$\varphi(c_t, m_t, k_t) = \ln\ c_t + \delta \ln [z_t \cdot a_t] + \beta \ln [(1-z_t) \cdot a_t]$$

$$\ln\ c_t + (\delta+\beta) \ln\ a_t + \delta \ln\ z_t + \beta \ln (1-z_t) \qquad (5)$$

We can then formulate the representative household's problem as the maximization of intertemporal welfare based on equation (5) subject to the flow budget constraint in equation (2). Thus, the household's problem is

$$\max_{c_t, z_t} \int_0^{\infty} [\ln\ c_t + (\delta+\beta) \ln\ a_t + \delta \ln\ z_t + \beta \ln (1-z_t)] e^{-\rho t} dt$$

$$\text{s.t.} \quad \dot{a}_t = [r_t a_t + w_t + x_t] - [c_t + (r_t + \pi_t + \sigma) z_t a_t].$$

Here consumption $c_t$ and real money balances $m_t$ in terms of per capita resources $a_t$, that is, $z_t$ are the control variables, and $a_t$ is the state variable. The household takes the paths of the real interest rate, the wage rate, the inflation rate and government transfers $[r_t, w_t, \pi_t, x_t]_{t=0}^{+\infty}$ and the (constant) policy parameter $\sigma$ as given. Recall that $n_t = 0, \forall t$, (no population growth) has been assumed. Furthermore, the household takes as given his initial level of real resources, $a_0$.

Setting up the current-value Hamiltonian for this problem and denoting $\mu_t$ as the current-value costate variable[7] the necessary first-order conditions for this maximization problem is

$$\frac{1}{c_t} - \mu_t = 0 \qquad (6)$$

$$\frac{\delta}{z_t} - \frac{\beta}{1-z_t} - \mu_t \cdot a_t (r_t + \pi_t + \sigma) = 0 \qquad (7)$$

$$-\left[\frac{\delta+\beta}{a_t} + r_t \mu_t - \mu_t (r_t + \pi_t + \sigma) \cdot z_t\right] = -\rho \mu_t + \dot{\mu}_t \qquad (8)$$

where we also require that equation (2) holds (with $n_t = 0$ ) and the transversality condition is satisfied, i.e.

or absolute wealth would not make a difference in the individual's decision and would yield similar results. As argued above I follow Plutarch here.

[7] For what is to follow we now use subscripts, except subscript t, to denote partial derivatives.

$$\lim_{t\to\infty} \mu_t \cdot a_t \cdot e^{-\rho t} = 0 \tag{9}$$

Recalling the definition of $z_t$ with $m_t \equiv z_t \cdot a_t$ and $k_t \equiv (1 - z_t) \cdot a_t$ and using equation (6) one can simplify equation (7) to

$$\frac{\delta}{z_t \cdot a_t} = \frac{\beta}{(1 - z_t) \cdot a_t} + \mu_t \cdot (r_t + \pi_t + \sigma)$$

$$\frac{\delta c_t}{m_t} - \frac{\beta c_t}{k_t} = (r_t + \pi_t + \sigma) \tag{10}$$

which implicitly describes the demand for real money balances $m$ as is shown below.

The equations (6) and (8) with $z_t = m_t/a_t$ entail that

$$\frac{\dot{c}_t}{c_t} = \frac{(\delta + \beta)c_t}{a_t} + r_t - \frac{(r_t + \pi_t + \sigma)m_t}{a_t} - \rho \tag{11}$$

whereby consumption growth depends on the "love of wealth" and the preference for money holdings. Unlike in conventional models the stocks of money and physical capital, which feature in $a_t$, bear on the growth rate of consumption. Notice that in this model there is, in general, a wedge between the real interest rate $r_t$ and the time preference rate $\rho$. It is not difficult to see that in a steady state when $\dot{c}_t = 0$ the real interest rate $r_t$ will in general not be equal to the time preference rate.

Using equation (10) where

$$\frac{\delta c_t}{m_t} - \frac{\beta c_t}{k_t} = (r_t + \pi_t + \sigma)$$

the expression for the consumption growth rate in equation (11) boils down to

$$\begin{aligned}\frac{\dot{c}_t}{c_t} &= \frac{(\delta + \beta)c_t}{a_t} + r_t - \left(\frac{\delta c_t}{m_t} - \frac{\beta c_t}{k_t}\right)\frac{m_t}{a_t} - \rho \\ &= \frac{\beta c_t}{a_t} + \left(\frac{\beta c_t}{k_t}\right)\frac{m_t}{a_t} + r_t - \rho = \frac{\beta c_t}{a_t}\left[\frac{k_t + m_t}{k_t}\right] + r_t - \rho.\end{aligned}$$

Thus, the growth rate of consumption is given by

$$\frac{\dot{c}_t}{c_t} = \beta\left(\frac{c_t}{k_t}\right) + r_t - \rho \tag{12}$$

which shows that "love of wealth", i.e. $\beta$ is an important determinant of the consumption growth rate. In particular, if $\beta$ is zero, we are back to the conventional and simplest money-in-the-utility model, where the economy dichotomizes into a real and nominal sector. This is because, if that is the case, in steady state $r_t = \rho$. But here with a $\beta$ that is taken to be non-zero, consumption growth depends on how people value capital.

## 3 The long-run general equilibrium

We now focus on the conventional approach to let supply and demand forces interact in an equilibrating way with one another at each point in time. This changes some of the insights of Part I in important ways. In particular, now let the factor markets be -determined by marginal productivity considerations. It turns out that the depreciation of money has important implications for the accumulation of physical capital and the long-run position of, that is, the steady state of the economy.

In order to close the model for the long-run equilibrium we now put structure on policy. To that end assume that the new issuance of money $\dot{M}_t$ depends on a constant fraction $\theta$ of the outstanding stock of nominal money $M_t$, plus the cost to be borne by replacing the "rotten" money due to "rusting", that is, $\sigma M_t$. In this paper $\theta$ and $\sigma$ are (constant) policy variables of the monetary authority.

Thus, $\dot{M}_t = \theta M_t + \sigma M_t$ and so the (gross) issuance of money is $(\theta + \sigma)M_t$. Letting $d(M_t/P_t)/dt \equiv \dot{m}_t$ yields

$$\dot{m}_t = \frac{\dot{M}_t}{P_t} - \left(\frac{\dot{P}_t}{P_t}\right)\left(\frac{M_t}{P_t}\right)$$

Notice that in this model the gross issuance of money is irredeemable. Thus, issued bank notes cannot be exchanged back at the monetary authority.

As $m_t \equiv M_t/P_t$ one obtains $\dot{M}_t/P_t = \dot{m}_t + \pi_t m_t$ where $\pi_t \equiv \dot{P}_t/P_t$. But then $\frac{\dot{M}_t}{P_t} = \frac{(\theta+\sigma)M_t}{P_t} = (\theta + \sigma)m_t$ so that real money balances change according to

$$\dot{m}_t = (\theta + \sigma - \pi_t)m_t. \tag{13}$$

Thus, (real) money growth is determined by $\theta + \sigma - \pi_t$, where $\theta$ and $\sigma$ are controlled by the government.

Furthermore, the monetary authority raises seigniorage by its issuance of money $(\theta + \sigma)M_t$ which, in this representative agent economy, is rebated lump-sum and in real terms to the household by assumption. In the quotes presented at the end of the paper, you find that Gesell argues that the rebating of seigniorage is governed by law. Notice that he also harbored the idea that the seignorage on (paper) money depreciation should be burnt in an oven. Here, in turn, I follow the standard assumptions in the monetary economics literature by which the seigniorage is completely related to the private sector. Thus, $x_t = (\theta + \sigma)m_t$.

In order to obtain clear-cut results the analysis is now restricted to satisfy the following criteria.

**Assumption 2** The aggregate technology is Cobb-Douglas and given by $Y = F(K, N) = K^{\alpha}N^{1-\alpha}$ where $0 < \alpha < 1$. Thus, $y = f(k) = k^{\alpha}$ where $y = Y/N$ and $k = K/N$.

**Assumption 3** Firms are price takers and maximize profits.

These assumptions form the basis for the marginal productivity theory of factor remuneration to hold in the ensuing analysis.

**Definition 1** A long-run general equilibrium consists of paths for consumption, real money balances, nominal money balances, the physical capital stock, the nominal interest rate, the real interest rate, the inflation rate, and government transfers $[c_t, m_t, M_t, k_t, i_t, r_t, \pi_t, x_t]_{t=0}^{+\infty}$ such that

1. money demand is described by equation (7);
2. investment decisions satisfy equation (8);
3. the transversality (9) condition is satisfied;
4. households obey their budget constraints (2)
5. the supplied real money balances are irredeemable and evolve according to $\dot{m}_t = (\theta + \sigma - \pi_t)m_t$, where $\theta$ and $\sigma$ are determined by the monetary authority;
6. the lump-sum transfers to the household are equal to the seigniorage from the money issue so that $x_t = (\theta + \sigma)m_t$;
7. the production uses a constant return to scale technology, and firms maximize profits where $w_t = f(k_t) - r_t k_t$ and $r_t = f'(k_t)$ and the factor markets clear;
8. the money and asset markets clear;
9. prices are flexible and monetary policy works, i.e. the money supply is exogenous.

We continue to focus on the factor rewards to labor, $w_t$, and capital, $r_t$ to measure distribution. This is compatible with Gesell's reasoning as can reasonably be inferred from his writings.

It is important to notice that now the marginal productivity theory of distribution holds (Definition 1, point 7). That rules out negative real interest in any long-run equilibrium, i.e. a situation when the market eventually clears. As the present model is of the standard variety, this also holds at any point in time given the concurrent optimizing behavior of the agents. It is not clear whether Gesell was thinking along those lines, though. But here we are interested in the validity of his hypotheses in the light of contemporaneous modeling.

The general equilibrium satisfying the definition above is characterized by a system of one static and three differential equations. In particular, the equilibrium is described by the differential equation (13)

$$\frac{\dot{m}_t}{m_t} = \theta + \sigma - \pi_t$$

where $\theta + \sigma = \pi$ in steady state, and the differential equation equation (12)

$$\frac{\dot{c}_t}{c_t} = \beta\left(\frac{c_t}{k_t}\right) + r_t - \rho$$

as well as the dynamic budget constraint in equation (2) with $n_t = 0$,

$$\dot{a}_t = [r_t \cdot a_t + w_t + x_t] - [c_t + (r_t + \pi_t + \sigma) \cdot m_t] \text{ and } a_t \equiv k_t + m_t$$
$$\dot{k}_t + \dot{m}_t = [r_t \cdot (k_t + m_t) + w_t + x_t] - [c_t + (r_t + \pi_t + \sigma) \cdot m_t]$$

In equilibrium, seigniorage revenue paid out to the household is $x_t = (\theta_t + \sigma)m_t$ . Furthermore, $\dot{m}_t = (\theta_t + \sigma - \pi_t)m_t$ as well as $r_t k_t + w_t = y_t = f(k_t) = k_t^\alpha$ hold in equilibrium. Thus, the last dynamic equation becomes

$$\dot{k}_t + (\theta_t + \sigma - \pi_t) \cdot m_t = [r_t \cdot (k_t + m_t) + w_t + (\theta_t + \sigma) \cdot m_t] - [c_t + (r_t + \pi_t + \sigma) \cdot m_t]$$
$$\frac{\dot{k}_t}{k_t} = \frac{f(k_t)}{k_t} - \frac{c_t}{k_t} - \sigma \cdot \frac{m_t}{k_t}$$

Finally, the static optimality condition in equation (10) requires

$$\frac{\delta c_t}{m_t} = \frac{\beta c_t}{k_t} + (r_t + \pi_t + \sigma) \Leftrightarrow \pi_t = \frac{\delta c_t}{m_t} - \frac{\beta c_t}{k_t} - (r_t + \sigma) \qquad (14)$$

We drop the time subscript from now on when it is clear that a variable depends on time, and index variables in steady state by *.

If we substitute the expression of $\pi$ from the last equation into the expression for the growth rate of real money balances one verifies that the equilibrium is characterized by a system of three dynamic equations.

$$\frac{\dot{k}}{k} = \frac{f(k)}{k} - \frac{c}{k} - \sigma \cdot \frac{m}{k} \tag{15a}$$

$$\frac{\dot{c}}{c} = \beta\left(\frac{c}{k}\right) + r - \rho \tag{15b}$$

$$\frac{\dot{m}}{m} = \theta + \sigma - \frac{\delta c}{m} + \frac{\beta c}{k} + (r + \sigma) \tag{15c}$$

where $f(k) = k^{\alpha}$ and $r = f'(k) = \alpha k^{\alpha-1}$.

Consequently, the steady state where $\dot{k} = \dot{m} = \dot{c} = 0$ is given by $\pi^* = \theta + \sigma$ and

$$f(k^*) = c^* + \sigma \cdot m^*, \tag{16a}$$

$$\beta\left(\frac{c^*}{k^*}\right) = \rho - r^*, \tag{16b}$$

$$\frac{\delta c^*}{m^*} = \frac{\beta c^*}{k^*} + (r^* + \theta + 2\sigma). \tag{16c}$$

Clearly, equation (16b) only makes sense if $\rho > r^*$, that is, when there is "love of wealth", and so $\beta > 0$. As the rate of time preference is an essentially unobservable variable, assume that indeed $\rho > r^*$. Below it will be shown that $\rho$ may not have to assume extremely unreasonable values to satisfy the condition. See footnote 12.

### 3.1 Steady State Analysis

Now we turn to the comparative static properties of the steady state. One readily verifies that

$$\pi^* = \theta + \sigma$$

Hence, with flexible prices $\pi$ adjusts so that the equality holds in a steady state. That means inflation is determined by the money growth rate $\theta$ and the money depreciation rate $\sigma$, which are constant and under the control of the monetary authority.

From equation (16a) we get

$$\sigma m^* + c^* = f(k^*) = y^* = w^* + r^* k^* \tag{17}$$

because the technology features constant returns to scale so that factor payments exhaust output.

The left-hand side of this equation can be interpreted as expenditures and the right-hand side is the income of the household in a steady state. Again note that apart from expenditures on real consumption the household must also 'buy' stamps for maintaining the face value of money. That outlay is captured by the amount $\sigma m^*$.

From equation (16c) we then get that

$$\frac{\delta \cdot c^*}{m^*} = \frac{\beta \cdot c^*}{k^*} + (r^* + \pi^* + \sigma) \text{ i.e. } \frac{k^*}{m^*} = \frac{\beta}{\delta} + \frac{(r^* + \pi^* + \sigma) \cdot k^*}{c^* \cdot \delta} \tag{18}$$

and from equation (16b) it follows that in steady state

$$\frac{c^*}{k^*} = \frac{\rho - r^*}{\beta}$$

Substituting the last expression in equation (18) and rearranging implies

$$m^* = \left(\frac{\delta k^*}{\beta}\right)\left[\frac{\rho - r^*}{\rho + \theta + 2\sigma}\right] = \frac{\delta c^*}{\rho + \theta + 2\sigma} \tag{19}$$

which captures the demand for real money balances in a steady state.

From the budget constraint in steady state (17) we have $c^*/k^* = y^*/k^* - \sigma \cdot m^*/k^*$. Substituting for $m^*/k^*$ from equation (18) yields

$$\frac{c^*}{k^*} = \frac{y^*}{k^*} - \sigma \cdot \left[\frac{\beta}{\delta} + \frac{(r^* + \pi^* + \sigma) \cdot k^*}{c^* \cdot \delta}\right]^{-1}$$

Now invoke $y/k = k^\alpha/k = k^{\alpha-1} = \alpha/\alpha \cdot k^{\alpha-1} = r/\alpha$, which holds at any point in time, plus the result that $c^*/k^* = (\rho - r^*)/\beta$. Then we get

$$\frac{\rho - r^*}{\beta} = \frac{r^*}{\alpha} - \sigma \cdot \left[\frac{\beta}{\delta} + \frac{(r^* + \pi^* + \sigma)}{\delta} \cdot \frac{\beta}{\rho - r^*}\right]^{-1}$$

$$\frac{\rho - r^*}{\beta}\left[\frac{\beta}{\delta} + \frac{(r^* + \pi^* + \sigma)}{\delta} \cdot \frac{\beta}{\rho - r^*}\right] = \frac{r^*}{\alpha}\left[\frac{\beta}{\delta} + \frac{(r^* + \pi^* + \sigma)}{\delta} \cdot \frac{\beta}{\rho - r^*}\right] - \sigma$$

$$\frac{\rho - r^*}{\delta} + \frac{r^* + \pi^* + \sigma}{\delta} + \sigma = \frac{r^*}{\alpha}\left[\frac{\beta}{\delta} + \frac{(r^* + \pi^* + \sigma)}{\delta} \cdot \frac{\beta}{\rho - r^*}\right]$$

$$\rho + \pi^* + \sigma + \delta\sigma = \frac{r^*}{\alpha}\left[\beta + (r^* + \pi^* + \sigma) \cdot \frac{\beta}{\rho - r^*}\right]$$

The last equation can be rearranged to yield

$$\rho + \pi^* + (1 + \delta) \cdot \sigma = \frac{r^*}{\alpha} \cdot \beta \cdot \left[\frac{\rho - r^* + r^* + \pi^* + \sigma}{\rho - r^*}\right] = \frac{r^*}{\alpha} \cdot \frac{\beta}{\rho - r^*} \cdot [\rho + \pi^* + \sigma].$$

For convenience rearrange the last expression to obtain

$$\Delta = \beta \ \text{ where } \ \Delta \equiv \left(1 + \frac{\delta\sigma}{\rho + \pi^* + \sigma}\right) \cdot \frac{\rho - r^*}{r^*} \cdot \alpha \ \text{ and } \ \pi^* = \theta + \sigma \tag{20}$$

which implicitly defines the capital stock in steady state, that is, $k^*$, as a function of the model's parameters, that is, $k^* = k^*(\sigma, \beta, \delta, \rho, \theta, \alpha)$.

From that, we obtain an important result. If $\sigma = 0$, then $k^*$ would be independent of monetary variables and the model would dichotomize into a monetary and real sector. To see this consider equation (20) to find that $k^*$ would then be independent of $\theta$ and $\sigma$. Furthermore, given that, $c^*$ and $y^*$ would also be independent of $\sigma$ and $\theta$.[8]

In contrast, if $\sigma$ is non-zero, then one easily verifies that the steady state capital stock depends on the money growth rate $\theta$ and the money depreciation rate $\sigma$. Thus, the model is then not super-neutral. [9]

**Proposition 1** *Without a Gesell tax, that is, when $\sigma = 0$, the model's steady state dichotomizes into a monetary and real sector. Monetary variables would then be neutral and superneutral in a long-run equilibrium. In contrast, if $\sigma \neq 0$, the model implies non-superneutrality.*

For the rest of the paper assume that $\sigma$ is non-zero. The economy does not dichotomize in that case and has, in general, a non-superneutral long-run equilibrium. As a consequence, the model features some form of a Mundell-Tobin effect.

Recall that Tobin (1965) and Mundell (1963) argued that monetary variables, in particular, realized or expected inflation, may have an effect on the real variables, especially on the (long-run) real interest rate of an economy. The effect is usually taken to be positive because it is argued that higher inflation causes people to hold less money and more real capital. That would then imply a lower real interest rate. [10]

In this model $\pi^* = \theta + \sigma$ in the steady state which, according to equation (20), bears on $k^*$ and so the long-run real interest rate $r^*$. Thus, it is through $\theta$ and $\sigma$ that the model features

---

[8] As argued above the model also dichotomizes when $\beta=0$. This is the world that Rösl (2006) analyzed. Clearly, and rather unsurprisingly, neutrality and super neutrality are then a feature of such a model. For this reason, amongst others, a positive $\beta$ is one constitutional feature of the present model.

[9] Recall that non-neutrality implies that money supply variables bear on long-run real variables like the steady-state capital stock. Non-super neutrality means that the rate of money supply growth has an effect on real variables. See, for example, Ahmed and Rogers (1996) for a clarifying study of this issue.

[10] Fischer (1988), p. 296/7 explains where the differences in the respective contributions of Tobin and Mundell lie. See also Temple (2000) for a more recent literature survey on the interaction of inflation and economic growth.

Mundell-Tobin effects. However, the effects of $\theta$ and $\sigma$ will be shown to be different. When any (positive) change in the variables leads to a higher real interest rate, I call that a *reverse Mundell-Tobin* effect.

We now analyze the comparative static properties of the steady state values of $k, m$, and $c$, and other variables of interest. I analyze the effects on $k$ in more detail in the main text and present the derivation for the other variables in the appendix.

For a change in $\sigma$ on $k$ note that

$$\Delta_r = -\left(1 + \frac{\delta\sigma}{\rho + \pi^* + \sigma}\right) \cdot \frac{\alpha\rho}{(r^*)^2} < 0 \tag{21}$$

As $r = \alpha k^{\alpha-1}$ we have $r_k < 0$. But then $\Delta_k = \Delta_r \cdot r_k > 0$ by equation (20), where again subscripts denote partial derivatives.

Furthermore, it turns out that, if $\delta > 0$,

$$\Delta_\sigma = \left(\frac{\delta(\rho + \theta + 2\sigma) - 2\delta\sigma}{(\rho + \theta + 2\sigma)^2}\right) \cdot \left(\frac{\rho - r^*}{r^*}\right) \cdot \alpha > 0$$

Then we have that $\Delta_k \cdot dk + \Delta_\sigma \cdot d\sigma = 0$ has to hold from equation (20). But consequently, we get $dk/d\sigma = -\Delta_\sigma/\Delta_k < 0$, that is, a higher money depreciation rate implies a lower steady-state capital stock. Thus, households choose to hold less physical capital which implies some form of a reverse *Mundell-Tobin* effect. Higher $\sigma$ may require more outlays for money holdings. These more "expensive" money holdings also make it more costly to hold physical capital. Holding less capital, in turn, entails a higher long-run real interest rate $r^*$, that is, it makes physical capital more "expensive". Hence, raising $\sigma$ appears to 'destroy' long-run wealth, that is, it implies a smaller, long-run physical capital stock.

For the effect of "love of wealth" $\beta$ one easily verifies that $dk/d\beta = 1/\Delta_k > 0$ so that an increase in the 'love of wealth' raises the long-run capital stock.

Valuing monetary transactions more (larger $\delta$ ) implies

$$\Delta_\delta = \left(\frac{\sigma}{\rho + \theta + 2\sigma}\right) \cdot \frac{\rho - r^*}{r^*} \cdot \alpha > 0$$

so that $dk/d\delta = -\Delta_\delta/\Delta_k < 0$. Clearly, if people derived more utility from money transactions (higher $\delta$ ) they might wish to hold more money, but in the model they definitely want to have less physical capital, implying a higher real interest rate $r^*$.

The effect of more impatience (larger $\rho$ ) depends on

$$\begin{aligned}\Delta_\rho &= \frac{\alpha}{r^*}\left[1+\frac{\delta\sigma}{\rho+\theta+2\sigma}\right]-\alpha\left(\frac{\rho-r^*}{r^*}\right)\left[\frac{\delta\sigma}{(\rho+\theta+2\sigma)^2}\right]\\ &= \frac{\alpha}{r^*}\left[1+\frac{\delta\sigma}{\rho+\theta+2\sigma}-\frac{(\rho-r^*)\delta\sigma}{(\rho+\theta+2\sigma)^2}\right]\\ &= \frac{\alpha}{r^*}\left[1+\frac{\delta\sigma(\rho+\theta+2\sigma)-(\rho-r^*)\delta\sigma}{(\rho+\theta+2\sigma)^2}\right]=\frac{\alpha}{r^*}\left[1+\frac{\delta\sigma(\theta+2\sigma)+r^*\delta\sigma}{(\rho+\theta+2\sigma)^2}\right]>0\end{aligned}$$

so that $dk/d\rho = -\Delta_\rho/\Delta_k < 0$. Thus, when the representative household is more impatient, there will be less physical capital in steady state.

For the impact of the money growth rate $\theta$ I find

$$\Delta_\theta = -\left(\frac{\delta\sigma}{(\rho+\theta+2\sigma)^2}\right)\left(\frac{\rho-r^*}{r^*}\right)\cdot\alpha<0$$

from which it follows that $dk/d\theta = -\Delta_\theta/\Delta_k > 0$ so that $k^*$ would be larger.

Thus, with a positive money depreciation rate a higher money growth rate implies a positive *Mundell-Tobin* effect. This is because for a given positive $\sigma$ an increase in $\theta$ entails a higher steady state inflation rate $\pi^*$. But a higher $\theta$ has just been found to raise the long-run capital stock, coupled with a lower real interest rate. Therefore, this captures the main point of a positive *Mundell-Tobin* effect.

The parameter $\alpha$ represents the elasticity of output with respect to capital, but also the capital share, since it is assumed that firms are profit maximizers under conditions of perfect competition. As $r = \alpha k^{\alpha-1}$ we can express $\Delta = \beta$ in equation (20) as

$$\Delta = \left(1+\frac{\delta\sigma}{(\rho+\theta+2\sigma)}\right)(\rho-r^*)\cdot(k^*)^{1-\alpha} = \left(1+\frac{\delta\sigma}{(\rho+\theta+2\sigma)}\right)(\rho\cdot(k^*)^{1-\alpha}-\alpha)=\beta$$

Then it follows that

$$\Delta_\alpha = -\left(1+\frac{\delta\sigma}{(\rho+\theta+2\sigma)}\right)(\rho\cdot\ln\,k^*\cdot(k^*)^{1-\alpha})$$

which is negative as long as $\ln\,k^*$ is larger than zero. [11]I assume this to be true, because it only depends on mild theoretical assumptions and very plausible values for the capital-labor ratio,

[11] Note that $k^{1-\alpha} = e^{(1-\alpha)\ln\,k}$ and $dk^{1-\alpha}/d\alpha = -\ln\,k\cdot e^{(1-\alpha)\ln\,k} = -\ln\,k\cdot k^{1-\alpha}$.

often shown in the empirical literature. [12]As a consequence, $dk/d\alpha = -\Delta_\alpha/\Delta_k > 0$ so that a higher capital share implies a higher steady state capital stock.

Summarizing these findings, the model features the following properties of the steady-state capital stock

$$k^* = k^* \underset{(-)}{\sigma}, \underset{(+)}{\beta}, \underset{(-)}{\delta}, \underset{(-)}{\rho}, \underset{(+)}{\theta}, \underset{(+)}{\alpha} \Bigg). \tag{22}$$

Clearly as $y = f(k)$ is monotonically increasing in $k$, the properties of $k^*(\cdot)$ carry over to steady state output $y^* = f(k^*(\cdot))$, and - in our Cobb-Douglas world - also to the wage rate $w^* = f(k^*) - f'(k^*) \cdot k^* = (1-\alpha)f(k^*)$ and the real interest rate $r^* = f'(k^*)$. The latter immediately follows from assumption 2.

From equation (12) consumption in a steady state is given by

$$c^* = \left(\frac{\rho - r^*}{\beta}\right) \cdot k^* \tag{23}$$

In Appendix A. 1 the reaction of steady-state consumption is analyzed and found to be characterized by

$$c^* = c^* \left( \underset{(-)}{\sigma}, \underset{(+)}{\beta}, \underset{(-)}{\delta}, \underset{(-)}{\rho}, \underset{(+)}{\theta}, \underset{(+)}{\alpha} \right)$$

$$c^* = c^* \left( \underset{(-)}{\sigma}, \underset{(+)}{\beta}, \underset{(-)}{\delta}, \underset{(-)}{\rho}, \underset{(+)}{\theta}, \underset{(+)}{\alpha} \right)\Bigg) \tag{24}$$

Two results are noteworthy here. The monetary policy variables $\theta$ and $\sigma$ have opposite effects on steady-state consumption. A higher money depreciation rate lowers it, whereas a higher money growth rate raises it. This is probably less surprising if one notes that higher $\theta$ raises income and capital, but $\sigma$ does not. Actually, more money depreciation is 'bad' for capital as well as income, and it competes through money depreciation outlays with consumption. The second interesting finding is that more 'love of wealth' makes more consumption possible in a steady state. Even though higher $\beta$ may seem to be only conducive to more investment, it leads to more steady-state capital and income, making a higher level of steady-state consumption possible. A related finding is presented in Rehme (2017) and analyzed there in more detail.

From equation (19) the demand for real balances in a steady state is given by

---

[12] Clearly the model requires ρ>r* and so $\rho > \alpha(k^*)^{\alpha-1}$ and $k^* > (\alpha/\rho)^{1/(1-\alpha)}$ . Thus, as long as α is larger than ρ then the condition k^*>1 is met. It is conventionally assumed that α is around 1/3 and ρ<<0.33.

$$m^* = \frac{\delta c^*}{\rho + \theta + 2\sigma}$$

As $\nu \equiv c/m$ it follows that in steady state $\nu^*$ is increasing in $\sigma$ and $\theta$. In that sense, the short-run and long-run effects of monetary policy on the velocity of money are very similar.

Next, in Appendix A. 2 the reaction of steady-state real money balances is analyzed. The findings there can be summarized by

$$m^* = m^*(\underset{(-)}{\sigma}, \underset{(+)}{\beta}, \underset{(?)}{\delta}, \underset{(-)}{\rho}, , \underset{(?)}{\theta}, \underset{(+)}{\alpha}) \tag{25}$$

Interestingly, households hold less money in a steady state when money depreciation is increased. This is because a higher $\sigma$ implies a higher velocity of money so that households need to hold less money in a long-run equilibrium to conduct their monetary transactions.

In turn, the effect of $\theta$ is not unambiguously clear and depends on the parameter values of the model. If $\delta$ and/or $\sigma$ are sufficiently small, then a higher money growth rate is coupled with less money holdings, but a higher velocity of money.

From equations (22), (24), and (25) and the expression of the welfare function in equation (4) the reactions of the steady state variables and welfare to changes in the variables of interest here yield the following.

**Proposition 2** *Given everything else, the introduction of a positive, previously nonexistent Gesell tax, which is kept in place forever, implies a higher velocity of money $\mathrm{v}^*$, a lower capital stock $k^*$, lower consumption $c^*$, and less holdings of real money balances $\mathrm{m}^*$ and so lower welfare $\varphi^*(c^*, m^*, k^*)$ in steady state. The steady-state return on capital $r^*$ rises so that some form of a reverse Mundell-Tobin effect is present.*

Thus, when looking at the effect of money depreciation on long-run outcomes in isolation, it seems that it would be a 'bad' policy option to introduce a depreciation rate on money holdings. Only [GC1] is validated. However, the introduction of a Gesell tax may not be too 'bad' an option because of the following.

**Proposition 3** *Given everything else and conditional on a positive (possibly very small) Gesell tax, a higher rate of money growth $\theta$ that is kept in place forever, implies a Mundell-Tobin effect. The capital stock $k^*$, output $y^*$, consumption $c^*$, and the velocity of money $\mathrm{v}^*$ would be higher,the long-run real interest rate $r^*$ lower and the wage rate $\mathrm{w}^*$ higher. Steady-state real money balances $\mathrm{m}^*$ may be higher or lower, depending on the parameter values of the model. The effect on long-run welfare is in general not unambiguously clear. For sufficiently high values of $\sigma$ and/or $\delta$, an increase in $\theta$ may raise $k^*, c^*$ and $m^*$ and long-run welfare $\varphi^*(c^*, m^*, k^*)$.*

Those findings would lend clear support to the Gesell Conjectures 1, and 2, [GC1], and [GC2]. Note that the proposition requires a positive Gesell tax. The latter is, thus, a necessary condition for any Mundell-Tobin effect to work. In order to see this more clearly consider the effects of joint variations in $\sigma$ and $\theta$ on steady state $k^*$. They can be determined from the differential $\Delta_k \cdot dk + \Delta_\sigma \cdot d\sigma + \Delta_\theta \cdot d\theta = 0$ using equation (20). We know that $\Delta_k > 0$. Thus, the reaction of $k$ is, for example, positive, if $\Delta_k \cdot dk > 0$. But that requires that $-\Delta_\sigma \cdot d\sigma - \Delta_\theta \cdot d\theta > 0$, that is:

$$-\left(\frac{\delta(\rho+\theta+2\sigma)-2\delta\sigma}{(\rho+\theta+2\sigma)^2}\right) Q \cdot d\sigma + \left(\frac{\delta\sigma}{(\rho+\theta+2\sigma)^2}\right) Q \cdot d\theta > 0$$

where $Q = \alpha\left(\frac{\rho - r^*}{r^*}\right)$ and the expressions for $\Delta_i, i = \sigma, \theta$ follow from above. This holds if both $\sigma$ and $\theta$ are changed. Again we see that, if $\sigma$ is zero, $\theta$ does not affect $k^*$.

For a non-zero $\sigma$, and simultaneous changes in both policy variables simplification yields that a positive effect on steady-state $k$ is present if

$$\sigma \cdot d\theta > (\rho + \theta) \cdot d\sigma$$

As a higher $\sigma$ lowers $k^*$ whereas a higher $\theta$ raises it, the change in $\theta$ must be sufficiently strong, that is, it must obey $d\theta/d\sigma > (\rho+\theta)/\sigma$ to have an overall positive effect on $k^*$.

**Result 1** *In general monetary policy conducted through changes in σ and/or θ has ambiguous effects on the steady-state capital stock* $k^*$ *. If the relative changes in the two monetary policy variables satisfy* $d\theta/d\sigma > (\rho+\theta)/\sigma$*, that is, if the change in θ is sufficiently strong and positive, given that money depreciation is present or its change is positive and given, then the long-run capital stock* $k^*$ *can be increased and the long-run real interest rate* $r^*$ *decreased.*

Thus, by a right combination of $\sigma$ and $\theta$ the monetary authority can generate a Mundell-Tobin effect with a higher long-run physical capital stock and lower real interest rate. This appears to be in line with Gesell's idea that expansionary monetary policy increases real activity. Only here it is found that simply focusing on money depreciation alone may not be enough for generating a positive effect on real variables. Although a necessary condition in this model, money depreciation has to be coupled with (new) money creation, that is, it must be accompanied by the injection of "new money" into the economy to have any positive effect on real variables in the long run.

## 4 Transitional Dynamics

The dynamic system of the equations in (15) can be log-linearized in a standard way to yield insights into the transitional dynamics and convergence properties of the system. The technical details for that are presented in Appendix C.

As the dynamics of the system are essentially governed by the same variables as in the standard Sidrauski model, one can employ the same arguments as in, for example, Blanchard and Fischer (1989), Appendix B of chapter 4, and Fischer (1979).

Thus, note that the capital stock is given, but the money stock and consumption can jump at any point in time. As a consequence, if the system is to have a (locally) unique stable path, it must have two positive roots (or a pair of complex roots with positive real part) and one negative root. If that is the case, the jump variables take on (initial) values that make the system converge. The analysis of the roots that govern the speed of convergence of the system is presented in the appendix. For the present model that implies the following result.

**Proposition 4** *Given a positive money depreciation rate, an increase in $\sigma$ speeds up, and an increase in $\theta$ lowers the speed of convergence to the steady state.*

Interestingly, that is a complement of the result in Fischer (1979), who shows that more money growth would lead to faster convergence when the utility function is non-logarithmic and the steady state features asymptotic superneutrality.

In turn, in this paper, the presence of (positive) money depreciation entails that the steady state is non-superneutral, but convergence is slower if the money growth rate $\theta$ is increased and we have a *logarithmic* utility function. [13]

### 4.1 Numerical simulation

The model is calibrated along some commonly observed magnitudes. The resulting system is then solved for those values. As a starting value assume that the initial capital stock, which is a given (state) magnitude, is taking a value of 5, that is, by assumption $k_0 = 5$. For the other parameters of the model consider the following values.

Table 1: Simulation

| $\alpha$ | $\beta$ | $\delta$ | $\rho$ | $\theta$ | $\sigma$ |
|---|---|---|---|---|---|
| 0.33 | 0.1 | 0.02 | 0.1 | 0.01 | 0.01 |

[13] Recall that if σ=0, then r^* is independent of σ and θ. So the latter variables would not impinge on convergence in that case. A similar result for logarithmic utility can be found in Fischer (1979).

The major reason for working with these values is that they command wide support in the literature. For example, the value for $\alpha$ is pretty standard and that for $\delta$ is almost the same as in Walsh (2010), p. 72. The money growth rate implied by Walsh is roughly equivalent to $\theta = 0.01$ for quarterly U.S. data on money supply M1, but in the model here I take the sum of $\theta$ and $\sigma$ to equal the long-run inflation rate, which many people consider to be around two percent.

An exception may be the value of $\rho$ which is taken to be a lot higher than what is conventionally used in empirical work. However, when one reminds oneself that the time preference rate is an important and somehow pervasive, but, nevertheless, ultimately quite unobservable concept, I assume a value of 10 percent, because it will make the other calibrated values correspond to ranges one finds in the literature.

Furthermore, note that $\beta$ is also very difficult to measure. Even some data of the World Value Service are not clearly established to be good measures of the "love of wealth", although the latter has clearly been identified by hermeneutic thinking (e.g. in philosophy, psychology, history and sociology among others) to be an important deep fundamental for social and, particularly, economic relationships. Here I calibrate $\beta$ so the long-run interest rate assumes a reasonable value.

With that in mind, the parameter values generate the following steady state magnitudes of the variables of interest.

Table 2: Simulated Steady State Values

| $k^*$ | $y^*$ | $k^*/y^*$ | $r^*$ | $m^*$ | $k^*/m^*$ | $c^*$ | $v = c^*/m^*$ | $\pi^*$ |
|---|---|---|---|---|---|---|---|---|
| 9.016 | 2.081 | 4.332 | 0.077 | 0.320 | 28.200 | 2.078 | 6.500 | 0.020 |

These numbers imply a steady-state inflation rate $\pi^* = \sigma + \theta$ of two percent. The steady-state capital stock and output are then calculated as $k^* = 9.016$ and $y^* = 2.081$. [14]That implies a capital-output ratio of about four which seems realistic for many countries. Then the steady state (i.e. long-run) return on capital is about 7.7 percent, which is broadly in line with many findings in the literature. See, for example, Jordá, Knoll, Kuvshinov, Schularick, and Taylor (2017), Table 11, for recent evidence.

Furthermore, the implied velocity of money in circulation is around 6.5 for measures such as $v = c^*/m^*$ or $v_1 = y^*/m^*$ which one approximately finds as a period average, for example, for the United States for the period 1960-2015.

[14] The simulation and the numerical convergence analysis were carried out in MATHEMATICA. The code used for the results and graphs below is available upon request.

From equation (35) in the appendix, we get the following numerical representation of the calibrated, log-linearized system

$$\begin{pmatrix} d\ln\ k/dt \\ (d\ln\ c)/dt \\ (d\ln\ m)/dt \end{pmatrix} = \begin{pmatrix} 0.0769 & -0.2305 & -0.0004 \\ -0.0743 & 0.0230 & 0.0000 \\ -0.0744 & -0.1069 & 0.1300 \end{pmatrix} \times \begin{pmatrix} d\ln\ k \\ d\ln\ c \\ d\ln\ m \end{pmatrix} + \begin{pmatrix} -0.035d\sigma \\ 0 \\ 2d\sigma + 1d\theta \end{pmatrix}$$

where $d\sigma$ and $d\theta$, our variables of interest here, denote the differentials of $\sigma$ and $\theta$ which are constants.

From that one obtains

$$\begin{pmatrix} d\ln\ k \\ d\ln\ c \\ d\ln\ m \end{pmatrix} = \xi_1 \begin{pmatrix} -0.713 \\ -0.469 \\ -0.496 \end{pmatrix} e^{\lambda_1 \cdot t} + \begin{pmatrix} -0.045d\sigma + 0.0041d\theta \\ -0.144d\sigma + 0.0131d\theta \\ -15.529d\sigma - 7.6792d\theta \end{pmatrix} \tag{26}$$

as the solution to the system. The derivation can be found in Appendix C.1. Here $\lambda_1 = -0.084$ is the only negative root of the system for the given parameter values. Its associated eigenvector is $(-0.713, -0.469, 0.496)$ and $\xi_1$ is a constant that needs to be definitized.

As $c$ and $m$ are jump variables we concentrate on the first component of this system, i.e., the equation for the capital stock $k$ to determine the constant $\xi_1$ from initial conditions. Thus, we solve for $\xi_1$ when $t = 0$, that is, we solve

$$(d\ln\ k)_{t=0} = \ln\ k_0 - \ln\ k^* = \xi_1 \cdot (-0.713) \cdot e^{\lambda_1 \cdot 0} - (0.045) \cdot d\sigma + (0.0041) \cdot d\theta$$

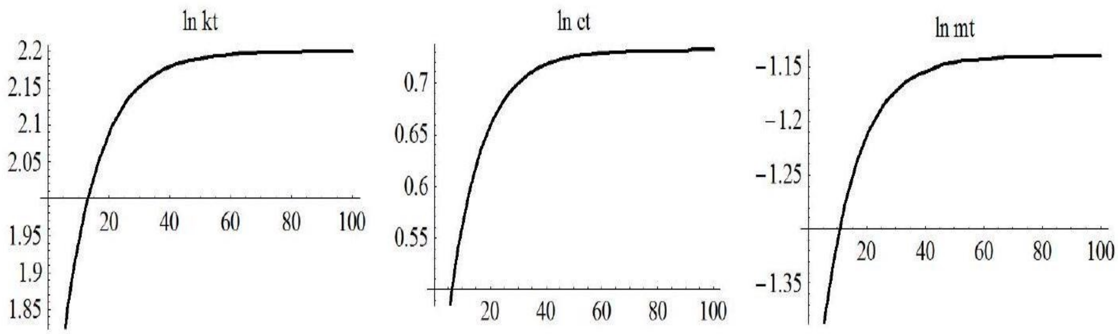
for $\xi_1$ with $e^{\lambda_1 \cdot 0} = 1$

This yields the definitized constant

$$\xi_1^* = \frac{(\ln\ k_0 - \ln\ k^*) + 0.045 \cdot d\sigma - 0.0041 \cdot d\theta}{-0.713} \tag{27}$$

where $k_0$ and $k^*$ are predetermined (non-jump) variables which are constant like the chosen values of $d\sigma$ and $d\theta$. Hence, $\xi_1^*$ is the constant sought after. Clearly, $\xi_1^*$ is also important for the paths of the jump variables $c$ and $m$ and it depends on $d\sigma$ and $d\theta$. The paths of $k_t, c_t$ and $m_t$ in natural logarithms are presented in the next figure, and those for the levels are presented in the appendix.

Figure 1: The paths of $k_t, c_t$ and $m_t$ in natural logarithms




We now conduct the following experiment for $\xi$ when each policy variable $\theta$ and $\sigma$ has a value of one percent so that the steady-state inflation rate is two percent, i.e. around $\sigma = \theta = 0.01$. The experiment is to increase the variables by one percentage point. For instance, we look at the system if $\sigma$ is raised from one to two percentage points, given $\theta$. The same is done for $\theta$. A final experiment is to consider a joint increase of one percentage point each, given that they were one percent.

Table 3: Changes in $\sigma$ and $\theta$ and the resulting $\xi_1^*$

| Case | $d\sigma$ | $d\theta$ | $\xi^*_{1\|\text{ Case}}$ |
|---|---|---|---|
| 0 | 0.00 | 0.00 | 0.827303 |
| 1 | 0.01 | 0.00 | 0.826675 |
| 2 | 0.00 | 0.01 | 0.827360 |
| 3 | 0.01 | 0.01 | 0.826732 |

The changes are taken around $\sigma = \theta = 0.01$.

From the table the differences are small. But it can be verified that

$$\xi^*_{1|2} > \xi^*_{1|0} > \xi^*_{1|3} > \xi^*_{1|1}$$

which one may have expected from the theoretical predictions.

First, consider the equation for the capital stock when $\lambda_1 = -0.084$. Given that $d\ln\, k = \ln\, k_t - \ln\, k^*$, we obtain from equation (26) that at any point in time $t$

$$\begin{aligned}\ln\, k_t - \ln\, k^* =& \xi_1^* \cdot (-0.713) \cdot e^{-0.084 \cdot t} - 0.045 \cdot d\sigma + 0.0041 \cdot d\theta \\ =& \left( \frac{(\ln\, k_0 - \ln\, k^*) + 0.045 d\sigma - 0.0041 d\theta}{-0.713} \right) (-0.713) e^{-0.084 \cdot t} \\ & -0.045 \cdot d\sigma + 0.0041 d\theta \\ \ln\, k_t =& (1 - e^{-0.084 \cdot t}) \ln\, k^* + e^{-0.084 \cdot t} \ln\, k_0 \\ & + (e^{-0.084 \cdot t} - 1)(0.045 \cdot d\sigma - 0.0041 \cdot d\theta).\end{aligned}$$

From these relationships, one readily obtains that for any $t > 0$

$$(\ln\, k_t)_{|2} > (\ln\, k_t)_{|0} > (\ln\, k_t)_{|3} > (\ln\, k_t)_{|1}$$

Hence, at a long-run equilibrium with $\sigma$ and $\theta$ at one percent each, an increase in $\sigma$, or $\theta$, or both implies that an isolated increase in $\sigma$ of one percentage point forever, given no change in $\theta$, leads to a lower path of capital at each point in time where $t > 0$ in comparison to the initial long-run equilibrium. An isolated increase in $\theta$, given no change in $\sigma$, in turn, implies a higher path of capital for each $t > 0$.

Thus, an increase in $\theta$ implies a higher steady-state capital stock, but that requires a positive (non-zero) $\sigma$. Note, however, that a simultaneous positive change in both variables is not necessarily augmenting capital as the values for $(\ln\, k_t)_{|3}$ reflect.

For consumption and the changes considered one obtains the following

$$(d\ln\, c)_{|i} = (\ln\, c_t)_{|i} - \ln\, c^* = \xi_{1|i}^* \cdot (-0.469) \cdot e^{-0.084 \cdot t} - 0.144 \cdot d\sigma + 0.0131 \cdot d\theta$$

where $i = 0,1,2,3$ reflects the changes of $d\sigma$ and $d\theta$ contemplated in Table 3 and where initial consumption $(c_0)_{|i}$ jumps to a value that satisfies this equation.

Calculating the differences $(d\ln\, c)_{|i} - (d\ln\, c)_{|0}$ for $i = 1,2,3$ then reveals that

$$(\ln\, c_t)_{|2} > (\ln\, c_t)_{|0} > (\ln\, c_t)_{|3} > (\ln\, c_t)_{|1}$$

So an increase in $\theta$ that is in place forever is 'good' for consumption at each point in time, but again requires a non-zero money depreciation rate $\sigma$. That also holds for initial consumption $c_0$.

On the other hand, a higher $\sigma$ entails that initial consumption is lower than the value of steady state consumption without money depreciation. Furthermore, no matter what initial consumption is, consumption at $t$ will decrease from its initial value. From that one also verifies that, if you keep $d\sigma > 0$ in place forever, the new long-run value of consumption is lower.

Next, turn to real money balances that are also a jump variable in this model. From the arguments above one readily gets that

$$
\begin{aligned}
(d\ln\, m)_{|i} &= (\ln\, m_t)_{|i} - \ln\, m^* \\
&= \xi^*_{1|i} \cdot (-0.469) \cdot e^{-0.084 \cdot t} - 15.529 \cdot d\sigma - 7.6792 \cdot d\theta
\end{aligned}
$$

Then it is not difficult to verify that

$$(\ln\, m_t)_{|0} > (\ln\, m_t)_{|2} > (\ln\, m_t)_{|1} > (\ln\, m_t)_{|3}$$

In a long-run equilibrium, the policy changes contemplated would, thus, imply less money holdings at each point in time.

The result is not difficult to justify because in the model an increase in $\theta$ and $\sigma$ increases the velocity of money $\nu$ and as a consequence, people want to hold fewer real money balances at each point in time.

Summarizing these effects for permanent policy changes yields the following:

**Result 2** *The model's simulation yields that for the policy experiments considered that at each point in time.*

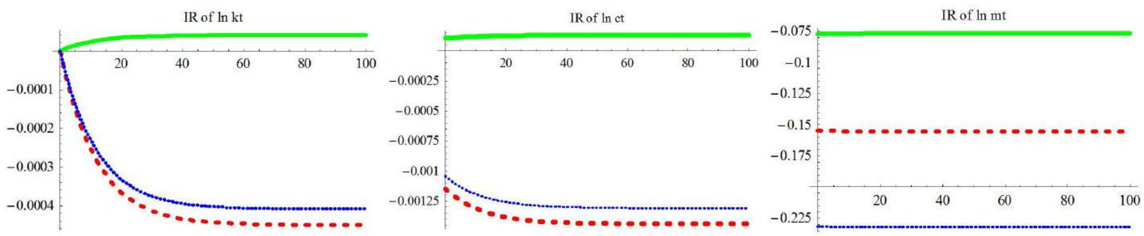
$$
\begin{aligned}
(\ln k_t)_{|2} &> (\ln k_t)_{|0} > (\ln k_t)_{|3} > (\ln k_t)_{|1} \\
(\ln c_t)_{|2} &> (\ln c_t)_{|0} > (\ln c_t)_{|3} > (\ln c_t)_{|1} \\
(\ln m_t)_{|0} &> (\ln m_t)_{|2} > (\ln m_t)_{|1} > (\ln m_t)_{|3}
\end{aligned}
$$

*The state variable $k$ as well as the jump variables $m$ and $c$ exhibit the same reactions for the policy changes at each (finite) point in time as the steady state reactions.*

Hence, for a given money depreciation rate $\sigma$ a higher money growth rate $\theta$ implies less money holdings (less monetization), more consumption, and generally a higher capital stock at each point in time. Lastly, note that the effects on transitional welfare are obvious. The following figure presents these effects as deviations from the path, where policy is not changed, that is, the paths presented in Figure 1.

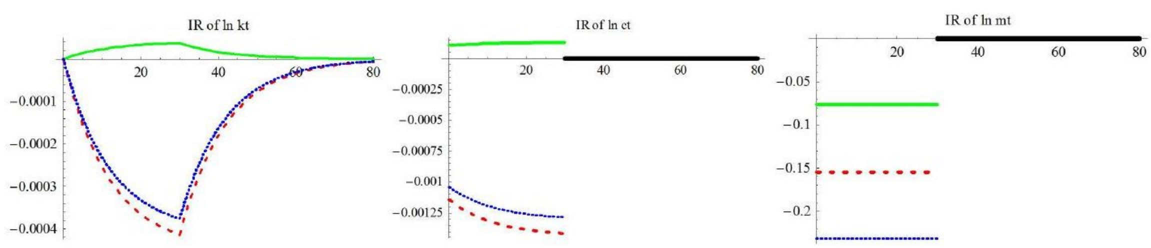
Furthermore, one verifies that temporary changes in $\theta$ and/or $\sigma$ produce the effects presented above. This is visualized in the following graph for a transitory change lasting 30 time periods. Again, the reactions are presented as deviations from the original path in Fig1. Thus, initial consumption and money holdings jump down after the changes involving $\sigma$, they jump up when $\theta$ is raised in isolation. They would then pursue a path getting to the new steady state if the policy changes were kept in place forever. But when the changes are transitory and revoked, consumption and money balances jump back to their pre-disturbance path. The natural logarithm of the state variable $k_t$ declines first and then converges to the unperturbed path after the changes involving changes in $\sigma$ are revoked. For isolated changes in $\theta$ (with no changes in $\sigma > 0$ ) these effects work in the opposite direction as is obvious from the graphs.

Figure 2: Permanent policy changes




Changes: $d\sigma$ - red dashed line, $d\theta$ - solid green line, $d\sigma + d\theta$ - dotted blue line

Figure 3: Short-run policy changes lasting the period $t \in [0,30]$




Changes: $d\sigma$ - red dashed line, $d\theta$ - solid green line, $d\sigma + d\theta$ - dotted blue line plotted as deviations from the benchmark log-linear model.

Lastly notice that, for example, a temporary drastic negative change in $\theta$ may well describe the Indian demonetization experience. Lower $\theta$ implies a lower capital stock, higher real interest rate, lower consumption, and lower real money according to the model. All this has more or less been observed in India but has been a temporary phenomenon. When remonetization finally got underway, $\theta$ was increased again and things operated in reverse. The open question is still whether the policy change has really been neutral for the Indian economy in the long run.

## 5 Conclusion

Long ago Silvio Gesell argued that money should 'rot' as any other good does and that depreciation of money (cash) in circulation would stimulate economic performance and be socially beneficial. He advocated a monetary system, which he called "free money", where fiat (paper) money would be legal tender and irredeemable.

In this paper, I question the claim that his ideas for an unconventional monetary policy cannot really be verified in modern economic theory frameworks. To this end I focus on four hypotheses Gesell made and analyze these using standard contemporaneous macroeconomic theory. The following findings of the paper are then noteworthy for a long-run environment.

First, it is shown that the steady state, that is, long-run inflation equals the money growth and depreciation rate. The economy dichotomizes into a monetary and real sector if there is no money depreciation. If the latter is present, the model features non-superneutrality. Money depreciation is a necessary condition for particular forms of a *Mundell-Tobin* effect.

Second, raising money depreciation in isolation lowers the steady state capital stock (wealth), consumption, income and welfare. It also implies a higher return to capital, but a lower steady-state wage rate. Thus, more money depreciation seems to destroy wealth and is not 'good' for labor. Higher money depreciation only implies a higher velocity of money.

Third, Gesell did not consider money depreciation as the only monetary policy tool. Here I find that, for a given positive money depreciation rate, an increase in the money growth rate produces a *Mundell-Tobin* effect. Thus, higher money growth increases steady-state inflation, but also the steady-state capital stock, output, and consumption. It implies a higher long-run wage rate and a lower return to capital. The consequences for the holdings of real money balances and so for total welfare are not unambiguously clear. But the velocity of money increases. However, the partial welfare channels through consumption and wealth work clearly in a positive direction. Hence, the conjectures are broadly validated for the long run in this model.

Fourth, the transitional dynamics reveal that the speed of convergence increases if money depreciation is raised, and decreases if the money growth rate is higher. In the present model, Fischer (1979) is complemented, because here the steady state generally features non-superneutrality, the utility function is logarithmic, and convergence is slower when the money growth rate increases.

A simulation exercise reveals that the response of key variables to permanent changes in the monetary policy variables is qualitatively the same in the transition as in the steady state. That also holds for the jump variables, namely, initial money holdings and consumption. Furthermore, for temporary changes in the policy variables, one obtains that qualitatively the temporary responses, again, basically equal those for the steady state.

Hence, in the present model-framework, most of Gesell's claims can be verified. In Part I was shown that in a short-run, demand-determined equilibrium all claims can verified. Here in Part II of the paper and for a long-run equilibrium two of his claims follow directly, and the other two indirectly, because money depreciation is a necessary condition for a positive *Mundell-Tobin* effect. This may warrant the renewed interest in Gesell's ideas and their significance for long-run economic policy problems.

Of course, the analysis faces several caveats. The setup of the model is simple. Alternative utility and production functions might imply more complicated equilibria or the lack thereof. The introduction of fiscal policy may make the results less clean. 'Love of wealth' was captured by a constant. This begs the question of how changes over time in the 'love of wealth' may bear on the optimal paths. These and other extensions of the model are left for further research.

## Appendix

## A Comparative Statics

### A.1 The effects on steady state consumption $c^*$

Recall that in the steady state $c^* = \left(\frac{\rho - r^*}{\beta}\right) \cdot k^*$. The effects of $c^*$ are then determined as follows.

#### A.1.1 The sign of $dc^*/d\sigma$

Notice that

$$\frac{dc^*}{d\sigma} = \frac{1}{\beta}\left(-r_k^* \cdot \frac{dk^*}{d\sigma} \cdot k^* + (\rho - r^*) \cdot \frac{dk^*}{d\sigma}\right)$$

As $r_k^* < 0$ and $dk^*/d\sigma < 0$, it follows that $dc^*/d\sigma < 0$.

#### A.1.2 The sign of $dc^*/d\beta$

It is not difficult to verify that

$$\frac{dc^*}{d\beta} = \frac{1}{\beta}\left(-r_k^* \cdot \frac{dk^*}{d\beta} \cdot k^* + (\rho - r^*) \cdot \frac{dk^*}{d\beta} - \frac{(\rho - r^*) \cdot k^*}{\beta}\right). \tag{28}$$

From equation (20) and (21) we know that

$$\Delta_k = \Delta_r \cdot r_k^* = -\frac{\alpha \cdot x \cdot \rho \cdot r_k^*}{(r^*)^2} \quad \text{where} \quad x \equiv 1 + \frac{\delta\sigma}{\rho + \pi + \sigma} \tag{29}$$

Furthermore $dk/d\beta = 1/\Delta_k = 1/(\Delta_k \cdot r_k^*)$. Then equation (28) can be rearranged as

$$\begin{aligned}\frac{dc^*}{d\beta} &= \frac{dk^*/d\beta}{\beta}\left[-r_k^* \cdot k^* + (\rho - r^*) - \frac{(\rho - r^*) \cdot k^*}{\beta \cdot (dk^*/d\beta)}\right] \\ &= \frac{dk^*/d\beta}{\beta}\left[-r_k^* \cdot k^* + (\rho - r^*) - \frac{(\rho - r^*) \cdot k^*}{\beta}\left(\frac{\alpha \cdot x \cdot \rho \cdot r_k^*}{(r^*)^2}\right)\right].\end{aligned}$$

It turns out that the expression in square bracket is positive by the following arguments. We have that $\beta = (x \cdot (\rho - r^*) \cdot \alpha)/r^*, r^* = \alpha(k^*)^{\alpha-1}$, and $r_k^* = \alpha(\alpha-1)(k^*)^{\alpha-2}$. Making the substitutions for $\beta$ and $r_k^*$ where appropriate above yields

$$\begin{gathered}
[-\alpha(\alpha-1)(k^*)^{\alpha-2} \cdot k^* + (\rho - r^*) \\
+(\rho - r^*) \cdot k\left(\frac{\alpha \cdot x \cdot \rho \cdot \alpha(\alpha-1)(k^*)^{\alpha-2}}{(r^*)^2}\right) \cdot \frac{r^*}{x \cdot (\rho - r^*) \cdot \alpha}\Bigg] \\
\Leftrightarrow \left[-\alpha(\alpha-1)(k^*)^{\alpha-1} + (\rho - r^*) + \left(\frac{\rho}{r^*}\right)\alpha(\alpha-1)(k^*)^{\alpha-1}\right].
\end{gathered}$$

When using the substitutions again one finds that the expression in square brackets boils down to

$$-(\alpha-1) \cdot r^* + (\rho - r^*) + \frac{\rho}{r^*} \cdot (\alpha-1) \cdot r^* = \alpha \cdot (\rho - r^*)$$

which is positive because $\rho > r^*$ in the model. Hence, $dc^*/d\beta > 0$.

### A.1.3 The signs of $dc/d\delta$ and $dc/d\rho$

Recall that $r_k^* < 0$ and $\rho > r^*$. Then it follows that

$$\frac{dc^*}{dj} = \frac{1}{\beta}\left(-r_k^* \cdot \frac{dk^*}{dj} \cdot k^* + (\rho - r^*) \cdot \frac{dk^*}{dj}\right) < 0 \text{ where } j = \delta, \rho$$

because $dk^*/dj < 0$ for $j = \delta, \rho$.

### A.1.4 The signs of $dc^*/d\theta$ and $dc^*/d\alpha$

Recall that $r_k^* < 0$ and $\rho > r^*$. Then it follows that

$$\frac{dc^*}{di} = \frac{1}{\beta}\left(-r_k^* \cdot \frac{dk^*}{di} \cdot k^* + (\rho - r^*) \cdot \frac{dk^*}{di}\right) > 0 \text{ where } i = \theta, \alpha$$

because $dk^*/di > 0$ for $i = \theta, \alpha$.

## A.2 The effects on steady state real money balances $m^*$

Recall $m^* = \frac{\delta c^*}{\rho + \theta + 2\sigma}$, the effects of which are then determined as follows:

**The sign of** $dm^*/d\sigma$.

$$\frac{dm^*}{d\sigma} = \frac{\delta(dc^*/d\sigma)(\rho + \theta + 2\sigma) - 2\delta c^*}{(\rho + \theta + 2\sigma)^2} < 0 \text{ because } \frac{dc^*}{d\sigma} < 0$$

**The sign of** $dm^*/d\beta$

$$\frac{dm^*}{d\beta} = \frac{\delta(dc^*/d\beta)}{\rho + \theta + 2\sigma} > 0 \text{ because } \frac{dc^*}{d\beta} > 0$$

**The sign of** $dm^*/d\delta$. I want to show that

$$\frac{dm^*}{d\delta} = \frac{c^* + \delta(dc^*/d\delta)}{\rho + \theta + 2\sigma} \gtreqless 0$$

To this end recall that

$$c^* = \left(\frac{\rho - r^*}{\beta}\right)k^* \text{ and } \frac{dc^*}{d\delta} = \frac{1}{\beta}\left(-r_k^* \cdot k^* \cdot \frac{dk^*}{d\delta} + (\rho - r^*) \cdot \frac{dk^*}{d\delta}\right)$$

and $-r_k^* \cdot k^* = (1-\alpha)r^*$. So we get

$$\frac{dm^*}{d\delta} = \frac{c^* + \delta(dc^*/d\delta)}{\rho + \theta + 2\sigma} = \frac{\left[(\rho - r^*)k^* + \delta[(1-\alpha)r^* + (\rho - r^*)]\frac{dk^*}{d\delta}\right]}{\beta(\rho + \theta + 2\sigma)}$$
$$= \frac{\left[(\rho - r^*)k^* + \delta[(\rho - \alpha r^*)]\frac{dk^*}{d\delta}\right]}{\beta(\rho + \theta + 2\sigma)}$$

where we know that

$$\frac{dk^*}{d\delta} = -\frac{\Delta_\delta}{\Delta_k} = -\frac{\left(\frac{\sigma}{\rho + \theta + 2\sigma}\right) \cdot \frac{\rho - r^*}{r^*} \cdot \alpha}{-\left(1 + \frac{\delta\sigma}{\rho + \pi + \sigma}\right) \cdot \frac{\alpha \cdot \rho}{(r^*)^2} \cdot r_k^*} = \frac{\sigma(\rho - r^*) \cdot (r^*/\rho)}{(\rho + \theta + (2+\delta)\sigma) \cdot r_k^*}$$

Making the substitution above yields

$$\left[(\rho - r^*)k^* + \delta[(\rho - \alpha r^*)]\left\{\frac{\sigma(\rho - r^*) \cdot (r^*/\rho))}{(\rho + \theta + (2+\delta)\sigma) \cdot r_k^*}\right\}\right] \cdot B \qquad (30)$$

where $B \equiv [\beta \cdot (\rho + \theta + 2\sigma)]^{-1} > 0$. Pulling out $(\rho - r^*)$ the expression in square brackets is positive, zero or negative if

$$k^* \gtreqless -\delta[(\rho - \alpha r^*)]\left\{\frac{\sigma \cdot (r^*/\rho)}{(\rho + \theta + (2+\delta)\sigma) \cdot r_k^*}\right\}. \quad (31)$$

As $r_k^* = (\alpha - 1) \cdot r^* \cdot (k^*)^{-1}$ the inequality boils down to

$$(1-\alpha)r^*k^* \gtreqless \frac{\delta[(\rho - \alpha r^*)] \cdot \sigma \cdot (r^*/\rho) \cdot k^*}{(\rho + \theta + (2+\delta)\sigma)}$$
$$(1-\alpha)r^*k^* \cdot (\rho + \theta + (2+\delta)\sigma) \gtreqless \delta \cdot [(\rho - \alpha r)] \cdot \sigma \cdot (r^*/\rho) \cdot k^*$$

No clear relationship can be established for this inequality. For example, if $\delta$ or $\sigma$ are very low $(\delta, \sigma \to 0)$, then the inequality is positive and $dm/d\delta > 0$ would follow. In turn, if, for example, $\sigma$ is very large (e.g. $\sigma \to \infty$ ) then $dm/d\delta > 0$ would be implied. Hence, the sign of $dm/d\delta$ is generally not unambiguously clear.

**The sign of** $dm^*/d\rho$.

$$\frac{dm^*}{d\rho} = \frac{\delta(dc^*/d\rho)(\rho + \theta + 2\sigma) - \delta c^*}{(\rho + \theta + 2\sigma)^2} < 0 \text{ because } \frac{dc^*}{d\rho} < 0$$

**The sign of** $dm^*/d\theta$. We have

$$\frac{dm^*}{d\theta} = \frac{\delta(dc^*/d\theta)(\rho + \theta + 2\sigma) - \delta c^*}{(\rho + \theta + 2\sigma)^2}$$

where the sign of that expression depends on the sign of $(dc^*/d\theta)(\rho + \theta + 2\sigma) - c^*$ for non-zero $\delta$, and

$$c^* = \left(\frac{\rho - r^*}{\beta}\right)k^*, \frac{dc^*}{d\theta} = \frac{1}{\beta}\left(-r_k^* \cdot k^* \cdot \frac{dk^*}{d\theta} + (\rho - r^*) \cdot \frac{dk^*}{d\theta}\right) = \left(\frac{\rho - \alpha r^*}{\beta}\right)\frac{dk^*}{d\theta} \text{ and}$$
$$\frac{dk^*}{d\theta} = -\frac{\Delta_\theta}{\Delta_k} = \frac{\left(\frac{\delta\sigma}{(\rho + \theta + 2\sigma)^2}\right) \cdot \frac{\rho - r^*}{r^*} \cdot \alpha}{-\left(1 + \frac{\delta\sigma}{\rho + \theta + 2\sigma}\right) \cdot \frac{\alpha \cdot \rho}{(r^*)^2} \cdot r_k^*} = \frac{\delta\sigma(\rho - r^*) \cdot r^*k^*}{(\rho + \theta + (2+\delta)\sigma) \cdot \rho \cdot (1-\alpha)r^*}$$

where again I have used that $-r_k^* \cdot k^* = (1-\alpha)r^*$. Making the appropriate substitutions yields after simplification that the sign of $(dc^*/d\theta)(\rho + \theta + 2\sigma) - c^*$ depends on whether

$$\left(\frac{(\rho - r^*)k^*}{\beta}\right)\left[\frac{(\rho - \alpha r^*)\delta\sigma(\rho + \theta + 2\sigma)}{(\rho + \theta + (2+\delta)\sigma)\rho(1-\alpha)} - 1\right] \gtreqless 0.$$

The sign of the expression in square bracket depends on the model's parameters. For example, if $\sigma$ or $\delta$ are sufficiently small, the expression in square brackets is negative, if they are sufficiently large, it is positive. Hence, the sign of $dm^*/d\theta$ is not unambiguously clear.

## B Long-run welfare effects

Long-run period welfare is given by $\varphi^*$ and by equations (19) and (23) amounts to

$$\begin{aligned}\varphi^* =&\ln\ c^* + \delta\ln\ m^* + \beta\ln\ k^* \\ =&\left(\ln\left(\frac{\rho - r^*}{\beta}\right) + \ln\ k^*\right) \\ &+\delta\left(\ln\left(\frac{\delta}{\rho + \theta + 2\sigma}\right) + \ln\left(\frac{\rho - r^*}{\beta}\right) + \ln\ k^*\right) + \beta\ln\ k^*.\end{aligned}$$

Collecting terms then reveals that long-run period welfare is

$$\varphi^* = (1 + \delta + \beta)\ln\ k^* + (1 + \delta)\ln\left(\frac{\rho - r^*}{\beta}\right) - \delta\ln\left(\frac{\rho + \theta + 2\sigma}{\delta}\right)$$

where $v = \frac{\rho+\theta+2\sigma}{\delta}$ equals the velocity of money. Notice it has a negative effect on long-run welfare in this model.

### B.1 The effect of $\sigma$ and $\theta$

As $c^*, m^*$ and $k^*$ all depend negatively $\sigma$ if follows that $d\varphi^*/d\sigma < 0$.

For the money growth rate $\theta$ one calculates

$$\frac{d\varphi^*}{d\theta} = (1 + \delta + \beta) \cdot \frac{dk^*}{d\theta} \cdot \frac{1}{k^*} + (1 + \delta)\left(-r_k \cdot \frac{\beta}{\rho - r^*}\right) \cdot \frac{dk^*}{d\theta} - \frac{\delta}{\rho + \theta - 2\sigma}$$

From the main text, we know that

$$\frac{dk^*}{d\theta} = -\frac{-\left(\frac{\delta\sigma}{(\rho + \theta + 2\sigma)^2}\right)\left(\frac{\rho - r^*}{r^*}\right) \cdot \alpha}{-\left(1 + \frac{\delta\sigma}{\rho + \theta + 2\sigma}\right)\left(\frac{\alpha\rho}{(r^*)^2}\right) \cdot r_k}$$

I want to check whether $\frac{d\varphi^*}{d\theta} > 0$. This boils down to analyze whether

$$\left(\frac{\delta\sigma}{(\rho+\theta+2\sigma)^2}\right)\left(\frac{\rho-r^*}{r^*}\right)\cdot\alpha\cdot\left[(1+\delta+\beta)\cdot\frac{1}{k^*}+(1+\delta)\left(-r_k\cdot\left(\frac{\beta}{\rho-r^*}\right)\right)\right]>$$
$$\left(\frac{\delta}{\rho+\theta-2\sigma}\right)\cdot(-1)\cdot\left(1+\frac{\delta\sigma}{\rho+\theta+2\sigma}\right)\left(\frac{\alpha\rho}{(r^*)^2}\right)\cdot r_k.$$

Cancellation by common terms then yields

$$\sigma\cdot(\rho-r^*)\cdot\left[(1+\delta+\beta)\cdot\frac{1}{k^*}+(1+\delta)\left(-r_k\cdot\left(\frac{\beta}{\rho-r^*}\right)\right)\right]>$$
$$(\rho+\theta+(2+\delta)\sigma)\left(\frac{\rho}{r^*}\right)\cdot(-r_k).$$

Note that $-r_k=\alpha(\alpha-1)k^{\alpha-2}=(\alpha-1)\cdot r^*\cdot(k^*)^{-1}$. Substituting this in and rearrangement yields

$$\sigma\cdot(\rho-r^*)\cdot\left[(1+\delta+\beta)+(1+\delta)\left((1-\alpha)\cdot r^*\cdot\left(\frac{\beta}{\rho-r^*}\right)\right)\right]>$$
$$(\rho+\theta+(2+\delta)\sigma)\cdot\rho\cdot(1-\alpha).$$

It is not difficult to see that the last inequality does not always hold and depends in an important way on the parameters of the model. For example, if $\sigma$ is very low, it does not hold. It may hold for sufficiently large values of it, though. It may also hold if $\beta$ is sufficiently large. But in general, no clear overall relationship between $w^*$ and $\theta$ holds.

But it is definitely so that the partial effects of $\theta$ on welfare through the consumption and capital channel raise welfare derived from them, that is, they raise welfare conditionally. In the model, the impact of the velocity of money and its reaction to changes in $\theta$ are so large that the other partial effects are outweighed.

## C Analysis of the Transitional Dynamics

The dynamic system of the equations in (15) can be formulated in (natural) logs as

$$\frac{d\ln\,k}{dt}=e^{(\alpha-1)\ln\,k}-e^{\ln\,(c/k)}-\sigma e^{\ln\,(m/k)} \tag{32a}$$
$$\frac{d\ln\,c}{dt}=\beta e^{\ln\,(c/k)}+\alpha e^{(\alpha-1)\ln\,k}-\rho \tag{32b}$$
$$\frac{d\ln\,m}{dt}=\theta+2\sigma-\delta e^{\ln\,(c/m)}+\beta e^{\ln\,(c/k)}+\alpha e^{(\alpha-1)\ln\,k} \tag{32c}$$

In steady state $\frac{d\ln k}{dt} = \frac{d\ln c}{dt} = \frac{d\ln m}{dt} = 0$ so that

$$e^{(\alpha-1)\ln k^*} = e^{\ln(c^*/k^*)} + \sigma e^{\ln(m^*/k^*)} \qquad (33a)$$

$$\beta e^{\ln(c^*/k^*)} + \alpha e^{(\alpha-1)\ln k^*} = \rho, \qquad (33b)$$

$$\delta e^{\ln(c^*/m^*)} = \theta + 2\sigma + \beta e^{\ln(c^*/k^*)} + \alpha e^{(\alpha-1)\ln k^*}. \qquad (33c)$$

From these equations, it then follows that in a steady state

$$e^{\ln(c^*/m^*)} = \frac{\rho+\theta+2\sigma}{\delta} \text{ and } \sigma e^{\ln(m^*/k^*)} = e^{(\alpha-1)\ln k^*} - e^{\ln(c^*/k^*)} = \frac{r^*}{\alpha} - \frac{\rho - r^*}{\beta}$$

where $f(k^*)/k^* = (k^*)^{\alpha-1} = r^*/\alpha$ and

$$r^* = \alpha(k^*)^{\alpha-1} = \alpha e^{(\alpha-1)\ln k^*} \qquad (34)$$

Now we linearize the system in (32) to get

$$\begin{pmatrix} \frac{d\ln k}{dt} \\ \frac{d\ln c}{dt} \\ \frac{d\ln m}{dt} \end{pmatrix} = \Delta \times \begin{pmatrix} d\ln k \\ d\ln c \\ d\ln m \end{pmatrix}$$

where $d\ln j = \ln j - \ln j^* = \ln(j/j^*)$ for $j = k, c, m$, and the star * denotes variables that are in their steady state. Δ is defined as

$$\mathbf{\Delta} \equiv \begin{pmatrix} \Delta_{1k} & \Delta_{1c} & \Delta_{1m} \\ \Delta_{2k} & \Delta_{2c} & \Delta_{2m} \\ \Delta_{3k} & \Delta_{3c} & \Delta_{3m} \end{pmatrix}_{k^*,c^*,m^*}$$

and represents the Jacobian of the system, evaluated in steady state equilibrium. Its elements are given by

$$\Delta_{1k} = (\alpha-1)e^{(\alpha-1)\ln k^*} + e^{\ln(c^*/k^*)} + \sigma e^{\ln(m^*/k^*)}, \quad \Delta_{1c} = -e^{\ln(c^*/k^*)}, \quad \Delta_{1m} = -\sigma e^{\ln(m^*/k^*)},$$

$$\Delta_{2k} = -\beta e^{\ln(c^*/k^*)} + \alpha(\alpha-1)e^{(\alpha-1)\ln k^*}, \quad \Delta_{2c} = \beta e^{\ln(c^*/k^*)}, \quad \Delta_{2m} = 0,$$

$$\Delta_{3k} = -\beta e^{\ln(c^*/k^*)} + +\alpha(\alpha-1)e^{(\alpha-1)\ln k^*}, \quad \Delta_{3c} = -\delta e^{\ln(c^*/m^*)} + \beta e^{\ln(c^*/k^*)}, \quad \Delta_{3m} = \delta e^{\ln(c^*/m^*)}.$$

Using the information about the steady state values yields the following:

$$\Delta_{1k} = (\alpha-1)e^{(\alpha-1)\ln k^*} + e^{\ln (c^*/k^*)} + \sigma e^{\ln (m^*/k^*)}$$
$$= \alpha e^{(\alpha-1)\ln k^*} - e^{(\alpha-1)\ln k^*} + e^{\ln (c^*/k^*)} + \sigma e^{\ln (m^*/k^*)} = r^*$$

because $e^{(\alpha-1)\ln k^*} = e^{\ln (c^*/k^*)} + \sigma e^{\ln (m^*/k^*)}$ in steady state and $\alpha e^{(\alpha-1)\ln k^*} = r^*$.

$$\Delta_{1c} = -e^{\ln (c^*/k^*)} = \frac{r^*-\rho}{\beta} < 0$$

On account of equations (33b) and (34). Furthermore,

$$\Delta_{1m} = -\sigma e^{\ln (m^*/k^*)} = -e^{(\alpha-1)\ln k^*} + e^{\ln (c^*/k^*)} = -\frac{r^*}{\alpha} + \frac{\rho - r^*}{\beta} = \frac{\rho - r^*\left(1+\frac{\beta}{\alpha}\right)}{\beta} < 0,$$

i.e. for a positive money depreciation rate $\Delta_{1m}$ is negative. [15]

Next, we have

$$\Delta_{2k} = -\beta e^{\ln (c^*/k^*)} + \alpha(\alpha-1)e^{(\alpha-1)\ln k^*} = -(\rho - r^*) + (\alpha-1)r^* = \alpha r^* - \rho < 0$$
$$\Delta_{2c} = \beta e^{\ln (c^*/k^*)} = \rho - r^* > 0 \text{ and } \Delta_{2m} = 0.$$

For the effect on money growth, we get

$$\Delta_{3k} = -\beta e^{\ln (c^*/k^*)} + +\alpha(\alpha-1)e^{(\alpha-1)\ln k^*} = \Delta_{2k} = \alpha r^* - \rho < 0$$
$$\Delta_{3c} = -\delta e^{\ln (c^*/m^*)} + \beta e^{\ln (c^*/k^*)} = -\delta\left[\frac{\rho+\theta+2\sigma}{\delta}\right] + \rho - r^* = -(r^*+\theta+2\sigma)$$
$$\Delta_{3m} = \delta e^{\ln (c^*/m^*)} = \rho+\theta+2\sigma.$$

All of this and the definition $\Delta$ imply that the log-linearized system is given by

$$\begin{pmatrix} (d\ln k)/dt \\ (d\ln c)/dt \\ (d\ln m)/dt \end{pmatrix} = \begin{pmatrix} \Delta_{1k} & \Delta_{1c} & \Delta_{1m} \\ \Delta_{2k} & \Delta_{2c} & \Delta_{2m} \\ \Delta_{3k} & \Delta_{3c} & \Delta_{3m} \end{pmatrix}_{k^*,c^*,m^*} \times \begin{pmatrix} d\ln k \\ d\ln c \\ d\ln m \end{pmatrix}$$
$$= \begin{pmatrix} r^* & \frac{r^*-\rho}{\beta} & \frac{\rho - r^*\left(1+\frac{\beta}{\alpha}\right)}{\beta} \\ \alpha r^* - \rho & \rho - r^* & 0 \\ \alpha r^* - \rho & -(r^*+\theta+2\sigma) & \rho+\theta+2\sigma \end{pmatrix} \times \begin{pmatrix} d\ln k \\ d\ln c \\ d\ln m \end{pmatrix}.$$

[15] Note that for $\sigma = 0$, that is, when the model dichotomizes we would, of course, have $\Delta_{1m} = 0$.

In order to analyze the question of how the log-linearized system reacts to a change in monetary policy, that is, to changes in the Gesell Tax and the money growth rate one verifies that the complete linearized system is really given by [16]

$$\begin{pmatrix} (d\ln\, k)/dt \\ (d\ln\, c)/dt \\ (d\ln\, m)/dt \end{pmatrix} = \Delta \times \begin{pmatrix} d\ln\, k \\ d\ln\, c \\ d\ln\, m \end{pmatrix} + \begin{pmatrix} \frac{\rho - r^* \left(\frac{\beta+\alpha}{\alpha}\right)}{\sigma\beta} \\ 0 \\ 2 \end{pmatrix} \times d\sigma + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times d\theta \qquad (35)$$

where $d\sigma$ and $d\theta$ are scalars that represent the differential of $\sigma$ and $\theta$, respectively, and the entries of the column vector $v$ represent the response of the (log-linearized) differential system to a change in $\sigma$ when the transpose of $v$ is given by $v' \equiv \left(\frac{\rho - r^*\left(\frac{\beta+\alpha}{\alpha}\right)}{\sigma\beta}, 0{,}2\right)'$ and in steady state $\Delta_{1\sigma} = e^{\ln(m^*/k^*)}$ and $\sigma e^{\ln(m^*/k^*)} = \frac{r^*}{\alpha} - \frac{\rho - r^*}{\beta}$. [17]

In turn, the entries of the column vector $w$ represent the response of the (log-linearized) differential system to a change in $\theta$ when the transpose of $w$ is given by $w' \equiv (0{,}0{,}1)'$

We can then express the system in (35) in compact form as

$$\mathbf{J}' = \Delta\boldsymbol{J} + \mathbf{g} \text{ where } \mathbf{J}' = \begin{pmatrix} \frac{d\ln\, k}{dt} \\ \frac{d\ln\, c}{dt} \\ \frac{d\ln\, m}{dt} \end{pmatrix}, \mathbf{J} = \begin{pmatrix} d\ln\, k \\ d\ln\, c \\ d\ln\, m \end{pmatrix}, \text{ and}$$

$$\mathbf{g} = \begin{pmatrix} \frac{\rho - r^* \left(\frac{\beta+\alpha}{\alpha}\right)}{\sigma\beta} \\ 0 \\ 2 \end{pmatrix} \times d\sigma + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times d\theta = \begin{pmatrix} \frac{\rho - r^* \left(\frac{\beta+\alpha}{\alpha}\right)}{\sigma\beta} d\sigma \\ 0 \\ 2d\sigma + d\theta \end{pmatrix}.$$

This is a nonhomogeneous differential equation system. The homogeneous part is $\mathbf{J}' = \boldsymbol{\Delta J}$ and depends in an important way on the Jacobian $\boldsymbol{\Delta}$. In turn, the term g makes the system nonhomogeneous.

[16] Here the assumption is, of course, that the initial values are close to the steady state. Although log-linear approximations are widely used in macroeconomics, the requirement that they apply only as approximations in the neighborhood of the steady state can be regarded as disadvantage. See, for example, Barro and Sala-i-Martin (2004), p. 111.

[17] Again note that this only holds for a non-zero $\sigma$. If $\sigma = 0$, then in view of equation (19) we would have $v' \equiv \left(\frac{\delta}{\beta}\left[\frac{\rho - r^*}{\rho+\theta+2\cdot 0}\right], 0{,}2\right)'$ and then there is no effect of a change in $\theta$ on $k$ in the transition.

First, we solve the homogeneous part $\mathbf{J}' = \boldsymbol{\Delta J}$, that is $\mathbf{J}' - \boldsymbol{\Delta J} = \mathbf{0}$, by employing the guess $\mathbf{J} = \boldsymbol{x}e^{\lambda t}$. From this, we get [18]

$$\mathbf{J}' = \lambda \boldsymbol{x} e^{\lambda t} = \boldsymbol{\Delta x} e^{\lambda t}, \text{ hence } \lambda \boldsymbol{x} = \boldsymbol{\Delta x}.$$

For a nontrivial solution, we need the eigenvalues (roots) and the eigenvectors of this three-dimensional system. The general solution of the homogeneous system is given by

$$\boldsymbol{J_h} = \xi_1 \boldsymbol{x}^{(\mathbf{1})} e^{\lambda_1 \cdot t} + \xi_2 \boldsymbol{x}^{(\mathbf{2})} e^{\lambda_2 \cdot t} + \xi_3 \boldsymbol{x}^{(\mathbf{3})} e^{\lambda_3 \cdot t}.$$

For $i = 1,2,3$ the roots of the system are given by $\lambda_i$, the eigenvectors by $\boldsymbol{x}^{(i)}$, and the arbitrary constants by $\xi_i$.

As the dynamics of the system are essentially governed by the same variables as in the standard Sidrauski model, we can employ the same arguments as in, for example, Blanchard and Fischer (1989), Appendix B of chapter 4, and Fischer (1979). Hence, we note that the capital stock is given, but the money stock and consumption can jump at any point in time. As a consequence, if the system is to have a (locally) unique stable path, it must have two positive roots (or a pair of complex roots with positive real part) and one negative root. If this is the case the jump variables will take on (initial) values that make the system converge. [19]

In fact, we can determine this more rigorously for the present model by following arguments. It is well known that the product of the roots is equal to the determinant of $\boldsymbol{\Delta}$. Calculating the determinant then yields

$$\begin{aligned} |\Delta| = \lambda_1 \cdot \lambda_2 \cdot \lambda_3 &= \rho^2 r^* - \frac{\rho^2 r^*}{\alpha} + 2\rho r^* \sigma - \frac{2\rho^2 r^* \sigma}{\alpha} + \rho r^* \theta - \frac{\rho^2 r^* \theta}{\alpha} \\ &= -\frac{(1-\alpha)\rho r^* (\rho + 2\sigma + \theta)}{\alpha} < 0. \end{aligned}$$

Thus, either all three roots are negative, or there are two positive and one negative root. If the system features saddle path stability, the latter is true and we additionally should have that the trace of $\Delta$ which equals the sum of the eigenvalues be non-negative. The latter is easily calculated as

$$\text{tr}\,(\boldsymbol{\Delta}) = \lambda_1 + \lambda_2 + \lambda_3 = 2\rho + 2\sigma + \theta$$

[18] In this section I follow the solution method presented in Kreyszig (2006), ch. 4.

[19] If the system had, for example, three negative roots, then starting from any value of c and m. the system would - locally - converge. There would be nothing to tie down the money stock or the level of consumption c. See Blanchard and Fischer (1989), p. 204.

which is indeed positive. Hence, at least one root is positive. With $|\mathbf{\Delta}| < 0$ and tr $(\mathbf{\Delta}) > 0$ the system is, therefore, saddle-path stable, because for our $3 \times 3$ system at least one root is positive so that there can only be one negative root. In summary, there will be two positive and one negative eigenvalue in the system.

One can also calculate the eigenvalues of the system. They are given by

$$\lambda_1, \lambda_2, \lambda_3 = \left\{ -\frac{-\alpha\rho \pm \sqrt{\alpha\rho} \cdot \sqrt{\alpha\rho + 4(1-\alpha)r^*}}{2\alpha}, \rho + 2\sigma + \theta \right\}.$$

Let $\lambda_1$ denote the negative root. Given the parameters it satisfies

$$\lambda_1 = -\frac{-\alpha\rho + \sqrt{\alpha\rho} \cdot \sqrt{\alpha\rho + 4(1-\alpha)r^*}}{2\alpha} < 0.$$

It is important to note that the negative root governsthe speed of convergence of the system. The more negative the negative eigenvalue $\lambda_1$ is, the faster the speed at which the system converges to its steady state. In this context, it is not difficult to verify that $d\lambda_1/d\sigma < 0$ and $d\lambda_1/d\theta > 0$. That means that as you increase $\sigma$, the root $\lambda_1$ will be more negative and so the convergence to the steady state will be faster, whereas an increase in $\theta$ is associated with a less negative root, implying that convergence will be slower. From that Proposition 4 in the main text follows in a straightforward manner.

Notice that we cannot have a convergent system when any of the roots is positive and the associated eigenvector $x^{(i)}$ is non-zero. One way to rule out explosive paths is to set the arbitrary constant associated with a positive root equal to zero. In our context, $\lambda_1 < 0$, and $\lambda_2, \lambda_3 > 0$, and $\xi_1 \neq 0$, but then we need $\xi_2 = \xi_3 = 0$ to rule out explosive behavior. As a consequence, the solution to the homogenous system boils down to $\boldsymbol{J_h} = \xi_1 \boldsymbol{x}^{(1)} e^{\lambda_1 \cdot t}$.

For a particular solution of the nonhomogeneous system above and since the vector $\mathbf{g}$ is constant, we try a constant column vector $\boldsymbol{J_p} = \boldsymbol{a}$ with components $\alpha_1, a_2$ and $a_3$ [20] As a consequence, $J_p' = \mathbf{0}$ and substitution in the system $\mathbf{J}' = \Delta \boldsymbol{J} + \mathrm{g}$ yields $\Delta \boldsymbol{a} + \mathrm{g} = \mathbf{0}$. Solving for the components of a, we get the following system under the assumptions made so far

$$\boldsymbol{J} = \boldsymbol{J_h} + \boldsymbol{J_p} = \xi_1 \boldsymbol{x}^{(\mathbf{1})} e^{\lambda_1 \cdot t} + \mathbf{a}.$$

The last step then is to use the initial conditions to definitize the constant $\xi_1$. Let $\tilde{\xi}_1$ denote the definitzed constant and let $\tilde{\xi}_1 \cdot \boldsymbol{x}^{(1)} \equiv \tilde{\boldsymbol{x}}^{(1)}$. Then the solution of our system is given by

$$\boldsymbol{J} = \boldsymbol{J_h} + \boldsymbol{J_p} = \tilde{\boldsymbol{x}}^{(\mathbf{1})} e^{\lambda_1 \cdot t} + \mathbf{a}.$$

[20] In this paragraph I closely follow Kreyszig (2006), p. 133.

The numerical simulation below clarifies the procedure in more detail.

### C.1 Numerical simulation

From the values in Tables 1 and 2, one gets the following numerical representation of the system in (35),

$$\begin{pmatrix} \frac{d\ln\ k}{dt} \\ \frac{d\ln\ c}{dt} \\ \frac{d\ln\ m}{dt} \end{pmatrix} = \begin{pmatrix} 0.0769 & -0.2305 & -0.0004 \\ -0.0743 & 0.0230 & 0.0000 \\ -0.0744 & -0.1069 & 0.1300 \end{pmatrix} \times \begin{pmatrix} d\ln\ k \\ d\ln\ c \\ d\ln\ m \end{pmatrix} + \begin{pmatrix} -0.035d\sigma \\ 0 \\ 2d\sigma + 1d\theta \end{pmatrix}$$

where $d\sigma$ and $d\theta$, our variables of interest in this section, denote the differentials of $\sigma$ and $\theta$ which are constants. The numerical convergence analysis was carried out in MATHEMATICA. The code used for the results is available upon request.

The $3 \times 3$ matrix represents the Jacobian $\mathbf{\Delta}$. The roots $\lambda$ of the homogenous part $\boldsymbol{J_h}$ are given by $(-0.0837, 0.1300, 0.1838)$.

As outlined above I concentrate on the negative root and call it $\lambda_1$. Thus, $\lambda_1 = -0.084$. Associated with $\lambda_1$ is the eigenvector $(-0.713, -0.469, 0.496)$. Hence, the general solution for our system is given by [21]

$$\boldsymbol{J_h} = \xi_1 \boldsymbol{x}^{(1)} e^{\lambda_1 \cdot t} = \xi_1 \begin{pmatrix} -0.713 \\ -0.469 \\ -0.496 \end{pmatrix} e^{\lambda_1 \cdot t}$$

For the particular solution $\boldsymbol{J_p}$ one obtains

$$\boldsymbol{J}_p = \begin{pmatrix} -0.045d\sigma + 0.0041d\theta \\ -0.144d\sigma + 0.0131d\theta \\ -15.529d\sigma - 7.6792d\theta \end{pmatrix}$$

that solves $\Delta a + \mathrm{g} = 0$ when looking at changes in $\sigma$ and $\theta$. From that we get

$$\begin{pmatrix} \mathbf{d\ln\ \boldsymbol{k}} \\ \mathbf{d\ln\ \boldsymbol{c}} \\ \mathbf{d\ln\ \boldsymbol{m}} \end{pmatrix} = \boldsymbol{J} = \boldsymbol{J_h} + \boldsymbol{J_p} = \boldsymbol{\xi_1} \begin{pmatrix} \mathbf{-0.713} \\ \mathbf{-0.469} \\ \mathbf{-0.496} \end{pmatrix} \boldsymbol{e^{\lambda_1 \cdot t}} + \begin{pmatrix} \mathbf{-0.045}\boldsymbol{d\sigma} + \mathbf{0.0041}\boldsymbol{d\theta} \\ \mathbf{-0.144}\boldsymbol{d\sigma} + \mathbf{0.0131}\boldsymbol{d\theta} \\ \mathbf{-15.529}\boldsymbol{d\sigma} - \mathbf{7.6792}\boldsymbol{d\theta} \end{pmatrix}$$

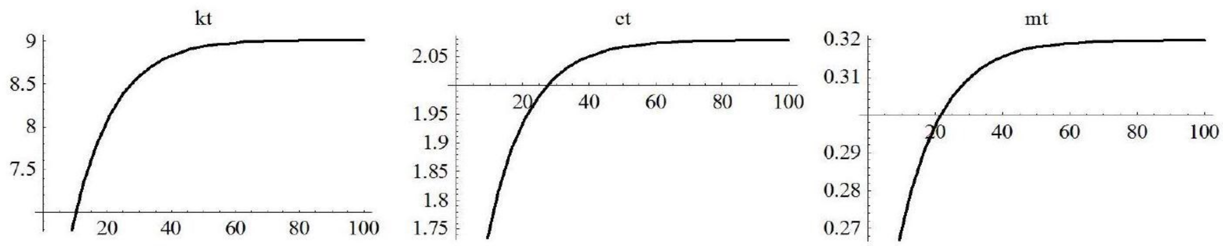
as the solution to the system which features in the main text as equation (26).

With the definitized constant $\xi_1^* = 0.827$ the graphs of the variables of interest in levels are presented in the following figure.

---

[21] Notice that with these values we get $|\Delta|$=-0.002 and tr($\Delta$)=0.23.

Figure 4: The paths of $k_t, c_t$ and $m_t$ in levels




## D Quotes [22]

## References

Ahmed, S., And J. H. Rogers (1996): "Long-Term Evidence on the Tobin and Fisher Effects: A New Approach," International Finance Discussion Papers 566, Board of Governors of the Federal Reserve System, Washington, D.C.

Bakshi, G. S., And Z. Chen (1996): “The Spirit of Capitalism and Stock-Market Prices," *American Economic Review*, 86, 133-157.

Barro, R. J., And X. SalA-I-Martin (2004): *Economic Growth*. MIT Press, Cambridge, Massachusetts, 2nd edn.

Blanchard, O. J., AND S. Fischer (1989): *Lectures on Macroeconomics*. MIT Press, Cambridge Mass.

Buiter, W. H. (2003): "Helicopter Money: Irredeemable Fiat Money and the Liquidity Trap," Working Paper 10163, NBER, Cambridge, Mass.

Carroll, C. D. (2000): "Why Do the Rich Save So Much?," in *Does Atlas Shrug? The Economic Consequences of Taxing the Rich*, ed. by J. B. Slemrod, pp. 465-484. Russel Sage Foundation and Harvard University Press, New York and Cambridge, Mass.

Darity, W. J. (1995): “Keynes' Political Philosophy: The Gesell Connection," *Eastern Economic Journal*, 21, 27-41.

Fisher, S. (1979): "Capital Accumulation on the Transition Path in a Monetary, Optimizing Model," Econometrica, 47, 1433-1439.

(1988): "Recent Developments in Macroeconomics," *Economic Journal,* 98, 294-339.

Gessel, S. (1920): *Die natürliche Wirtschaftsordnung. Rudolf Zitzmann Verlag*, available in English and translated by Philip Pye as *The Natural Order*, Peter Owen Ltd., London, 1958,

[22] Editor: The quotes are the same as in the first part of the paper, pages 125-129 in this issue.

4 edn., https://www. naturalmoney. org/NaturalEconomicOrder.pdf (last accessed on 26-01-2022).

Ilgmann, C., And M. Menner (2011): "Negative Nominal Interest Rates: History and Current Proposals," *International Economics and International Poli*cy, 8, 383-405.

Jordá, Ó., K. Knoll, D. Kuvshinov, M. Schularick, And A. M. Taylor (2017): "The Rate of Return on Everything, 1870-2015,' Discussion Paper 12509, CEPR, London.

Kreyszig, E. (2006): *Advanced Engineering Mathematics*. Wiley, Hoboken, N.J., 9th edn.

Kurz, M. (1968): "Optimal Economic Growth and Wealth Effects," *International Economic Review*, 9, 348-357.

Menner, M. (2011): “"Gesell Tax” and Efficiency of Monetary Exchange," Documento de Trabajo Working Paper WP-AD-2011-26, Instituto Valenciano de Investigaciones Económicas (IVIE), S.A., Alicante.

Mundell, R. (1963): “Inflation and Real Interest," *Journal of Political Economy*, 71, 280-283.

Pigou, A. C. (1941): *Employment and Equilibrium: A Theoretical Discussion*. Maxmillan, London.

Rehme, G. (2011): “'Love of Wealth' and Economic Growth," Darmstadt Discussion Papers in Economics (DDPIE) 209, Technische Universität Darmstadt, Darmstadt, Germany.

(2017): “'Love of Wealth' and Economic Growth,” Review of Development Economics, 21(4), 1305-1326.

(2018): “On 'Rusting' Money. Silvio Gesell’s Schwundgeld Reconsidered,” Darmstadt Discussion Papers in Economics (DDPIE) 233, Technische Universität Darmstadt, Darmstadt, Germany.

RösL, G. (2006): "*Regional Currencies in Germany - Local Competition for the Euro*?," Discussion Paper 43/2006, Deutsche Bundesbank, Frankfurt, Germany.

Sidrauski, M. (1967): "*Rational Choice and Patterns of Growth in a Monetary Economy*," American Economic Review, 57, 534-544.

Svensson, R., And A. Westermark (2016): "Renovatio Monetae: Gesell Taxes in Practice," Working Paper 327, Sveriges Riksbank, Stockholm.

Temple, J. (2000): "Inflation and Growth: Stories Short and Tall," *Journal of Economic Surveys*, 14, 395-426.

Tobin, J. (1965): “Money and Economic Growth," *Econometrica*, 33, 671-684.

Walsh, C. E. (2010): *Monetary Theory and Policy*. MIT Press, Cambridge, Massachusetts, 3rd edn.

Weber, M. (1930): *The Protestant Ethic and the Spirit of Capitalism*. Allen & Unwin, London, translated by Talcott Parsons edn.

Zou, H. (1994): “'The spirit of capitalism' and long-run growth*," European Journal of Political Economy*, 10, 279-293.