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Notes on the Limit of a Variable

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Notes on the Limit of a Variable

Respect for the reader's time and efforts compels us to state in unequivocal terms that the notion of limit as analysed here is, avowedly, of no interest whatsoever to the mathematician. We might add that it could even be repugnant to him. This qualification is not meant as a disparagement. Our purpose is not a mathematical one, nor is it a philosophical analysis of a mathematical notion for its own sake. Here we are interested in the notion of limit, particularly in the limit of a variable, in so far as it may contribute to a better understanding of certain philosophical positions that the contemporary scholastics all too promptly and too enthusiastically reject as utter nonsense. The failure to account for the truth concealed, though certainly present, in these positions is a most disturbing sign of philosophical stagnation.

In Book II of the *Physics* (c. 2., 185b25) Aristotle mentions the «more recent of the ancient thinkers» who tried to adjust oral expression to their doctrine of «the one». Some, like Lycophron, were led to omit the verbal copula «is», so that instead of saying «man is white» (where the interposition of «is» seems to destroy the unity of «man» and «white») they tried to express this unity adequately by saying merely «white man». However, this omission of a verb left them with an incomplete expression, which does not convey a full meaning. Because of this, others were led to change the mode of expression and say, instead of «white man», «man whitens». Saint Thomas explains why this seemed to solve their difficulty: «quia per hoc quod est *albari*, non intelligitur res aliqua, ut eis videbatur, sed quaedam subjecti transmutatio»[1]. They tried to overcome the distinct definiteness of «white» by substituting for it the dynamic, fluid «whiten» or «*becoming* white», with «white» itself never *given* and therefore never definitely opposed to the definite «man» who whitens, as one definite thing to another.

Aristotle's criticism of this view is wholly satisfactory as far as the ontological problem of «the one and the many» is concerned. «What 'is' may be many either in definition [for example 'to be white' is one thing, 'to be musical' another, yet the same thing may be both, so the one is many] or by division, as the whole and its parts. On this point, indeed, they were already getting into difficulties and admitted that the one was many— as if there was any difficulty about the same thing being both one and many, provided that these are not opposites; for 'one' may mean either 'potentially one' or 'actually one'»[2]. However, the problem of «the one and the many» has an epistemological scholium the detail of which has been largely supplied

1. *Ibid.*, lect. 4.
2. *Loc. cit.*, Ross transl.

by Neo-Platonism. There is also the problem of «the one and the many» on the part of the knower, when viewed in connection with the *means* of knowing[1]; for knowledge as such does not imply a one-to-one correspondence between «the one» or «the many» *known* and «the one» or «the many» *of the means of knowing.* In fact such a one-to-one relation between the natures known and their corresponding means of knowing is a property of the lowest form of intellectual knowledge. The very highest perfection of knowledge consists in knowing «the many» by one single concept or intelligible species. Although this doctrine is well known, we seem to have remained quite unaware of its relevance to the problems of modern philosophy.

Perhaps this statement calls for an illustration. We fully agree with Prof. Ernst Cassirer when he says: «The ideas of the One and the Many form the two poles about which all philosophic and religious thinking revolves»[2]. He has clearly stated the particular approach to this problem in the work of Nicholas of Cusa. Here is a quotation we would have the reader bear in mind throughout this present essay[3]:

> ...Both principles (of «*docta ignorantia*» and of «*coincidentia oppositorum*»), which had dominated theological thought for centuries, suddenly take a new turn in the fifteenth century. Their general significance is maintained; but they now receive a content of new problems and new interests. What had formerly been a negative principle of theology now becomes a positive principle of natural philosophy, cosmology, and epistemology. Nicholas Cusanus proceeds from his conception and interpretation of the idea of «*docta ignorantia*» to an acute criticism of the Aristotelian logic and the Aristotelian physics. Aristotle's logic is unexcelled in the precise working out of contradictions, in setting up the categories by which the classes of being are distinguished. But it is unable to overcome this opposition between the various classes of being; it does not press on to their real point of unification. Hence it remains caught in the empirical and the finite; it is unable to rise to a truly speculative interpretation of the universe. The physical universe of Aristotle is dominated by the opposition between «the straight» and «the curved»; motion in straight lines and motion in circles are for him essentially and radically distinct. But the transition to the infinitely large and the infinitely small shows that this is a matter not of an absolute but of a relative distinction. The circle with an infinite radius coincides with the straight line; the infinitely small arc is indistinguishable from its chord.

Here is a challenge to what we may call, for historical reasons, the Peripatetic mode of thinking. Scholastics must meet this challenge— Thomists in particular. The problem, even as we find it stated in Cassirer, is an old one. Prof. A. E. Taylor, in one of his most important contributions to the understanding of Plato[4], has given us definite proof of this; but, as we may gather from Cassirer's statement, it is also a con-

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1. See De Koninck's *Dialectique des limites comme critique de la raison*, in this same issue.

2. *Giovanni Pico della Mirandola.* A Study in the History of Renaissance Ideas, in Journal of the History of Ideas, April, 1942, Vol. III, No. 2, (Part I, pp. 123-144), p. 131.—Part II of this very important study appeared in the following issue, No. 3, pp. 319-346.

3. *Ibid.*, Part II, pp. 322-323

4. *Forms and numbers: a study in platonic metaphysics*, in his volume bearing the general title *Philosophical studies*, Macmillan and Co., London, 1943, pp. 91-150.

temporary problem. Furthermore, the passage we have just quoted cannot fail to remind us of Friedrich Engels' exclamation in *Dialectics of Nature:* «When the mathematics of straight and curved lines has thus pretty well reached exhaustion a new almost infinite field is opened up by the mathematics that *conceives curved as straight* [differential triangle] and *straight as curved* [curve of the first order with infinitely small curvature]. O metaphysics»[1]! Is it a challenge to metaphysical thought? It would be poor dialectic, to say the least, to counter this challenge with a blunt negation and to leave it at that.

It is only in the light of this problem that Hegel becomes intelligible. Although contemporary dialectical materialists, presumably, would not subscribe, without qualification, to Engels' ideas on the infinitesimal[2], it would be difficult to understand Stalin's characterization of the dialectical method without them. We refer to the following passage taken from his *Dialectical and historical materialism*[3]:

(a) Contrary to metaphysics, dialectics does not regard nature as an accidental agglomeration of things, of phenomena, unconnected with, isolated from, and independent of, each other, but as a connected and integral whole, in which things, phenomena, are organically connected with, dependent on, and determined by, each other.

The dialectical method therefore holds that no phenomenon in nature can be understood if taken by itself, isolated from surrounding phenomena, inasmuch as any phenomenon in any realm of nature may become meaningless to us if it is not considered in connection with the surrounding conditions, but divorced from them; and that, vice versa, any phenomenon can be understood and explained if considered in its inseparable connection with surrounding phenomena, as one conditioned by surrounding phenomena.

(b) Contrary to metaphysics, dialectics holds that nature is not a state of rest and immobility, stagnation and immutability, but a state of continuous movement and change, of continuous renewal and development, where something is always arising and developing, and something always disintegrating and dying away.

The dialectical method therefore requires that phenomena should be considered not only from the standpoint of their interconnection and interdependence, but also from the standpoint of their movement, their change, their development, their coming into being and going out of being.

The dialectical method regards as important primarily not that which at the given moment seems to be durable and yet is already beginning to die away, but that which is arising and developing, even though at the given moment it may appear to be not durable, for the dialectical method considers invincible only that which is arising and developing.

«All nature», says Engels, «from the smallest thing to the biggest, from a grain of sand to the sun, from the protista (the primary living cell—*Ed.*) to man, is in a constant state of coming into being and going out of being, in a constant flux, in a ceaseless state of movement and change.» (*Dialectics of Nature.*)

Therefore, dialectics, Engels says, «takes things and their perceptual images essentially in their interconnection, in their concatenation, in their movement, in their rise and disappearance.» (*Ibid.*)

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1. We quote the English translation annotated by J. B. S. Haldane, International Publishers, New York, 1940, p. 201.

2. See De Koninck's note on *Calcul et dialectique*, in this issue.

3. We quote from the text edited by International Publishers, New York, 1940, pp. 7-8.

Obviously this text must be understood in terms of the «rational kernel» of Hegelian dialectics. The metaphysics it refers to might stand for Hegel's Finite Understanding as opposed to Dialectical and Speculative Thought[1]. Certainly its meaning depends upon a very definite interpretation of the limit of a variable.

The present analysis of the notion of limit of a variable is undertaken for the purpose of exhibiting in simple form the ideas underlying the type of dialectic we have illustrated by quotation. The choice of this particular notion for analysis is not an arbitrary one, nor is it based on *our* interpretation of this type of dialectic, as is sufficiently clear from the texts cited. Actually all of Hegel's dialectic hinges upon this notion of limit, as can be seen from the following passage of his *Logic* (Encycl.), where he rejects the universality of the principle of contradiction:

> ...A notion, which possesses neither or both of two mutually contradictory marks, *e.g.* a quadrangular circle, is held to be logically false. Now though a multiangular circle and a rectilineal arc no less contradict this maxim, geomèters never hesitate to treat the circle as a polygon with rectilineal sides[2].

By what feat can this very ancient *mos geometrae* be twisted into an obvious instance of contradiction superseded?

If, in the course of our analysis, we relentlessly insist upon pointing out even the most tediously obvious, it is because we are not unmindful of the way in which distinguished philosophers have used the notion of limit of a variable and of what they have inferred from it. Witness the Marxian idea of social revolution. Even apart from this preoccupation, the type of analysis we shall indulge in would still reveal a complexity repugnant to the temperament of modern mathematical thinking. Mathematicians often do make statements of extreme importance to philosophy, but the value of such statements could be justified only by this type of analysis.

Descartes, finding fault with Aristotle's analysis of movement, wondered how a man could render so obscure what was apparently so simple. Perhaps, though, the obscurity was not where Descartes chose to assign it. As Professor Muirhead once said:

> ...It may be well to remind (the mathematicians) from the side of philosophy that here, as elsewhere, apparent simplicity may conceal a complexity which it is the business of somebody, whether philosopher or mathematician... to unravel.

1. When we inscribe a regular polygon within a circle, and then a larger one, and so on, to find the area of that circle, the purpose is presumably just that, namely, to find the area of the circle with ever increasing exactness. Montucla's observation is to the point:

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1. Compare this passage with Chapter VI of Hegel's *Logic* (Encycl.), transl. by W. Wallace, second edition, impression of 1931, pp. 143-155.

2. *Wallace* transl., p. 221.

«L'objet principal et primitif de la Géométrie est de mesurer les différentes espèces d'étendues que l'esprit considère; mais mesurer n'est autre chose que comparer une certaine étendue à une autre plus simple, et dont on a une idée plus claire et plus distincte. Partant de ce principe, les géomètres ont pris la ligne droite pour la mesure à laquelle ils rapporteraient toutes les longueurs; le quarré pour celle à laquelle ils rappelleraient les surfaces quelconques; le cube enfin pour celle des solides. Ainsi rectifier une courbe, quarrer une surface, cuber un solide, ne sont autre chose que déterminer leur grandeur, les mesurer. Quarrer un cercle n'est donc pas, comme l'imagine un vulgaire ignorant, faire un cercle quarré, ce qui est absurde; ou, comme semblent le croire certaines gens, faire un quarré d'un cercle; mais mesurer le cercle, le comparer à une figure rectiligne, comme au quarré de son diamètre, et connaître son rapport précis avec ce quarré ou enfin parce que l'un dépend de l'autre, déterminer le rapport de la circonférence avec le diamètre.» («On peut dire aussi», he adds in a footnote, «qu'il s'agit de construire un quarré dont la superficie soit égale à celle du cercle.»)[1]

However, the purpose of inscribing regular polygons is perhaps not necessarily confined to reaching the *area* of the circle. The purpose may also be to reach a clearer view of the *formal structure* of the circle:

...Le géomètre pur, Poincaré says...sans renoncer tout à fait au secours de ses sens,... veut arriver au *concept de la ligne sans largeur*, du point sans étendue. Il n'y peut parvenir qu'en regardant la ligne comme la limite vers laquelle tend une aire de plus en plus petite. Et alors, nos deux bandes, quelque étroites qu'elles soient, auront toujours une aire commune d'autant plus petite qu'elles seront moins larges et dont la limite sera ce que le géomètre pur appelle un point [2].

In this consideration, the purpose of viewing a term as a limit is not merely to reach more exact knowledge of the quantitative value of that term, but to obtain, somehow, a clearer concept of the proper form of the term. Montucla's «vulgaire ignorant» was perhaps too lightly disposed of, and the example of the «square circle» may have been a trifle too simple.

When Cassirer says that, contrary to what Aristotle held, the straight and the curved are essentially and radically not distinct, that the transition to the infinitely large and the infinitely small shows this to be a matter not of an absolute but of a relative distinction, that the circle with an infinite radius coincides with the straight line, and that the infinitely small arc is indistinguishable from its chord, he is very obviously referring to formal structure. Furthermore, there would otherwise be no point in presenting this view as going beyond Aristotle. Now, In Book VII of the Physics, where he determines the conditions of commensurability, Aristotle concludes that, if two things are to be fully commensurable, not only must they be without equivocation, but there must also be no difference either in that which is in the subject, or in the subject itself[3]. In the words of St. Thomas' commentary:

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1. *Histoire des recherches sur la quadrature du cercle* (1754), nouvelle édition revue et corrigée, Paris 1831, pp. 3-4.

2. *La science et l'hypothèse*, Paris, 1932, pp. 38-39. The italics are ours.

3. c. 4, 249a3:' Ἀλλ' ἆρα οὐ μόνον δεῖ τὰ συβλητὰ μὴ ὁμώνυμα εἶναι, ἀλλὰ καὶ μὴ ἔχειν διαφοράν μήτε ὃ μήτ' ἐν ᾧ.

...Oportet ea quae sunt comparabilia, non solum non esse aequivoca, quod erat primum; sed etiam non habere differentiam, neque ex parte subjecti primi in quo aliquid recipitur, quod erat secundum; neque ex parte ejus quod recipitur, quod est forma vel natura; et hoc est tertium [1].

This position is indeed the opposite of the one stated by Cassirer. For Aristotle, the coincidence and indistinction of two specifically different things is possible when we prescind from their difference, either formal or material. Straight and curved are indistinguishable when considered merely as lines. Or again, as St. Thomas adds in his commentary:

...Multa quidem secundum abstractam considerationem vel logici vel mathematici non sunt æquivoca, quæ tamen secundum concretam rationem naturalis ad materiam applicantis, æquivoce quodammodo dicuntur, quia non secundum eamdem rationem in qualibet materia recipiuntur.. [2].

In all these cases, commensurability is possible only because the differences of the terms compared have been dropped.

Now the question arises: what is absolute in the terms compared?—that which they have in common according to abstraction by confusion, or that in respect of which they are different and irreducible? When it is said that Aristotle's logic is inadequate, that it is unable to rise to a view of things in their absolute indistinction, does this mean that the depth of knowledge lies in the direction of growing confusion? This is obviously not what Cassirer means. The «truly speculative interpretation of the universe» would have to account for the irreducibles of the universe when relatively considered.

Nevertheless, we must face the consequence of the negation of Aristotle's conditions of commensurability, namely, that all things would be one according to their innermost nature: *τῷ λόγῳ ἕν τὰ ὄντα πάντα* [3]. In other words, our current irreducible definitions would have value only in the universe of experiential appearance, a universe comparable to Parmenides' phenomenal world of opinion. If such were the case, it is difficult to see how we could escape Aristotle's curt criticism of the latter's One.

...If all things are one in the sense of having the same definition, like 'raiment' and 'dress', then it turns out that they (Melissus and Parmenides) are maintaining the Heraclitean doctrine, for it will be the same thing 'to be good' and 'to be bad', and 'to be good' and 'to be not good', and so the same thing will be 'good' and 'not good', and man and horse; in fact, their view will be, not that all things are one, but that they are nothing; and that 'to be of such-and-such a quality' is the same as 'to be of such-and-such a size'[4].

2. Whether or not, or to what extent, this criticism may apply to the position presented by Cassirer is a matter we shall turn to later. We may all agree, however, that even the «speculative interpretation» would get nowhere if we did not presuppose the formal differences for example between straight and curved, whatever these differences might later turn out to be—essential and radical or merely relative. Hegel would have no quarrel with us on this point. If we are not mistaken, the following passage brings out the issue:

1. Lect. 7, n. 12.
2. *Ibid.*, n. 9.
3. *I Physic.*, c. 2, 185b19.
4. *Ibid.*, Ross transl.

...The merits and right of the mere Understanding (which sticks to fixity of characters and their distinctness from one another) should unhesitatingly be admitted. And that merit lies in the fact, that apart from Understanding there is no fixity or accuracy in the region either of theory or of practice.

Thus, in theory, knowledge begins by apprehending existing objects in their specific differences. In the study of nature, for example, we distinguish matters, forces, genera and the like, and stereotype each in its isolation. Thought is here acting in its analytic capacity, where its canon is identity, a simple reference of each attribute to itself. It is under the guidance of the same identity that the process in knowledge is effected from one scientific truth to another. Thus, for example, in mathematics magnitude is the feature which, to the neglect of any other, determines our advance. Hence in geometry we compare one figure with another, so as to bring out their identity. Similarly in other fields of knowledge, such as jurisprudence, the advance is primarily regulated by identity. In it we argue from one specific law or precedent to another: and what is this but to proceed on the principle of identity?
...Understanding, too, is always an element in thorough training. The trained intellect is not satisfied with cloudy and indefinite impressions, but grasps the objects in their fixed character: whereas the uncultivated man wavers unsettled, and it often costs a deal of trouble to come to an understanding with him on the matter under discussion, and to bring him to fix his eye on the definite point in question[1].

Turning now to the type of limit referred to by Cassirer, we agree that the constant [straight] and the variable [curved] must be formally, specifically distinct: they must differ by definition. Since we are here interested in formal structure, and not primarily in quantitative value, the movement toward the limit must exhibit a tendency of the variable toward the constant in its very difference. Here is indeed an inescapable paradox. On the one hand, the limit must be formally different, while on the other hand if, *per impossibile*, it could actually be reached in this difference, it would turn out to be of the species of the variable. Straight, say, would be a species of curve; circle, a species of polygon. The attainment of the limit would at the same time give us a species *of* polygon opposed *to* polygon as one species to another. Some polygon would be non-polygon. Whether or not this contradiction may be superseded, it is unquestionably a dead-lock to the Understanding.

I. THE VARIABLE ORDERED TO A LIMIT.

3. Let us analyse, in common terms, the variable x_n ordered to a limit, as to the meaning it has in the expression[2]

$$\lim_{n \to \infty} x_n = \xi$$

where x_n stands for the variable in its successive values according to the index $n \to \infty$ or $0, 1, 2, 3 \ldots$, and ξ for the limiting value. The sequence we are interested in, and which we suppose x_n to stand for, is an infinite series whose terms we denote by s_n, meaning:

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1. *Op. cit.*, pp. 144-145.

2. Cf. Konrad Knopp, *Theory and application of infinite series*, transl. Young, London, 1944.

$$s_0=a_0;\ s_1=a_0+a_1;\ s_2=a_0+a_1+a_2$$

Hence:

$$s_n=a_0+a_1+a_2+\ldots+a_n \qquad (n=0,1,2,\ldots).$$

The sequence may be expressed by:

$$a_0+a_1+a_2+\ldots a_n+\ldots$$

or by:

$$a_0+a_1+a_2+\ldots$$

or by the more expressive symbol of an infinite series:

$$\sum_{n=0}^{\infty} a_n$$

The infinite series we are concerned with is a convergent series conforming to Cauchy's test, where, from some place on in the series $\sum a_n$ of positive terms, $a_n \leqq a^n$ with $o < a < 1$, that is

$$\sqrt[n]{a_n} \leqq a < 1.$$

As an instance we shall take the numerical series

$$\sum_{n=0}^{\infty} \frac{1}{2^n}$$

or:

$$1, 1+\tfrac{1}{2}, 1+\tfrac{1}{2}+\tfrac{1}{4}, 1+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}, \ldots$$

whose limit is 2. A corresponding geometrical example would be a series of regular inscribed polygons whose sides double to infinity and whose limit is a circle.

We must note too that we shall confine ourselves to pointing out those aspects of the variable which are relevant to the ideas some philosophers have associated with the notion of limit.

The type of variable we are concerned with has both unity and multiplicity, identity and otherness, form and matter, fixity and mobility. It has a unity (a) which can be seen in the common definition of its values. All regular polygons have one definition in common. The multiplicity (a) of the different values the variable may assume (three-sided, six-sided, twelve-sided, etc.) itself constitutes a unity (b) distinct from, though conditioned by, the unity (a) of what they have in common. There is a third unity (c) which embraces the multiplicity (c) constituted by the other two unities, (a) and (b). This unity (c) is a first indication of the proper unity of the variable.

The common definition remains identical (a), whatever the differences of the values may be[1]. It is identical for any value. That which is expressed by the common definition is other (a) than the limit, and hence any value it may be predicated of is other than the limit. Thus, many-sided, and any many-sided plane figure, is other than circle. There is also an otherness (b) intrinsic to the variable. Any value of the variable is, both individually and structurally, different from any other value of the variable. Each value is itself a distinct species and has a proper definition. This otherness (b) is other (c) than the otherness first mentioned (a). The importance of this distinction will be brought out later. However, the one and the other are opposed to the limit, although not in the same way. The identity (c) of the variable taken as a whole, embraces both the indentity of the common definition (a) and the identity (b) of each variable with itself, the variable as such being thereby opposed as other (d) to the limit. This otherness (d) is a first indication of the identity and otherness proper to the variable under consideration.

By the form (a) of the variable may be meant that which is expressed by the definition, as predicable of any one of its values; for, in the order of predication, that which is predicated is as form with respect to that of which it is predicated. This form is one and identical for any of the values. By the matter (a) of the variable may be meant the values of which the form (a) may be predicated. Thus, the multiplicity (a) and the otherness (b) are on the part of the matter. There is also form (b) on the part of the matter: each value has its own distinct form which is expressed by its proper difference, such as *three*-sided, or *six*-sided. Furthermore, there is a form (c) that comprises both the form first mentioned (a) and the many forms (b) which are the matter (a) of the form (a), now taken as the matter (b) of the proper form (c) of the variable as such.

In assuming its different values, the variable is both variable and invariable. Identity, as opposed to change, is an essential condition of change[2]. No matter what value the variable assumes, no matter how different or how many the forms (b), the form expressed by the definition and predicable of any one of its values (a) remains invariable (a). Unless we have identified polygon with one of its species, we cannot say that one species, no matter how many sides it has, is more polygon than the other, since «many-sided» embraces any number of sides. Now the

1. «...Respondetur: identitatem non esse formaliter idem quod unum; sed identitas unitatem supponit, et superaddit vel negationem mutationis, sicut dicitur manere idem quamdiu non mutatur seu alteratur a statu suo: vel relationem rationis ejusdem ad se ipsum, quatenus dicitur idem sibi et non identificari alii extra se.» John of St. Thomas, *Curs. Theol.*, Solesmes edit., T.II, p. 111, n. 32.

2. «Manifestum est enim quod omnis motus habet unitatem ab eo quod movetur: quia scilicet illud quod movetur est unum et idem manens in toto motu; et non est indifferenter id quod movetur, uno motu manente, quodcumque ens, sed illud idem ens quod prius incepit moveri: quia si esset aliud ens quod postea moveretur, deficeret primus motus, et esset alius motus alterius mobilis. Et sic patet quod mobile dat unitatem motui, quæ est eius continuitas. Sed verum est quod mobile est aliud et aliud secundum rationem. Et per hunc modum distinguit priorem et posteriorem partem motus: quia secundum quod consideratur in una ratione vel dispositione, cognoscitur quod quæcumque dispositio fuit in mobili ante istam signatam, pertinebat ad priorem partem motus; quæcumque autem post hanc erit, pertinebit ad posteriorem. Sic igitur mobile et continuat motum et distinguit ipsum.» *In IV Physic.*, lect. 18 n. 9.

variable embraces both the invariable form (a) and the variable matter (a). Because of the variation of its matter (a), variability (b) is attributed to the variable as a whole; because of the invariability of its form (a), the variable is at the same time invariable (b). Thus the same variable is other and other: «aliud et aliud ratione».

The simplicity of the symbol we employ to express a variable of this type is apt to conceal the maze of terms and relations it stands for, leading us to forget that this complexity, which we have so far analyzed only in part, is the very reason why we have to use a symbol[1]. For this reason we shall henceforth call *symbolic,* all those properties which belong to the variable as a whole.

4. Let us now consider the class of the variable. The multiple values of the variable constitute more than a mere class: they have a common definable form and their differences have a common principle[2]. We use the term *class* to emphasize the material multiplicity (a) and the unity (b) of the variable.

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1. A symbol, as De Koninck points out in his lectures on scientific methodology, lies somewhere in between the name (comprising the verb) and the infinite name. (Cf. St. Thomas *In I Perihermeneias,* lect. 4, n. 13; *II,* lect. 1, n. 3) St. Albert calls the symbols employed in logic transcendent terms, which signify everything and nothing: «...Ideo terminis utimur transcendentibus, nihil et omnia significantibus. *Nihil* dico, quia nullam determinant materiam. *Omnia* vero dico significantibus: quia omnibus materiis sunt applicabiles, sicut sunt *a,b,c.*» (*Priora Anal. I,* Tract. I, c. 9, (Vivès-Borgnet, p. 472b). If, as the nominalists thought, the term *being* stood for the collection of beings, the term would not be a name proper, but a symbol. Taken in this strict sense, symbols are not to be confused with abbreviations. (Cf. W. E. Johnson's *Logic,* Cambridge, 1922, Part II, p. 46). As someone has put it: «Les mathématiques sont l'art de donner le même nom à des choses différentes». Symbols, however, do not always definitely stand for a collection. They may be used to signify anything that cannot be, or is not yet, properly named, as in $x+2=3$.. Again, the number - symbol 2, say, may be used either as a symbol proper—as when we wish to signify by one term the collection $1+1$; or it may be used as an abbreviation-symbol—when *one* and *one* are not merely *two* ones, but *one* two.

2. We are well aware that the term class, aggregate, or group, is not restricted to the type of class where the members have a common definable form. Objects may be one by some purely extrinsic designation such as «object in John's back-yard,» under which designation we find the incongruous class of (a) a broken milk-bottle, (b) a stray cat, (c) John, etc. This type of class is one as a bundle or a heap. Like Democritus, most modern authors consider this most tenuous conceivable whole as the most fundamental type of class, to be used as the norm for any species of class. We define it as a whole whose actual parts are one in any way. (Cf. Aristotle, *VIII Metaph.,* c. 6, 1045a9). Cantor's definition of an aggregate (Menge) amounts to the same: «any collection into a whole (Zusammenfassung zu einem Ganzen) M of definite and separate objects *m* of our intuition or thought.» (*The theory of transfinite numbers,* Jourdain transl., LaSalle, Illinois, 1941, p. 85). We agree that this is the most general kind of class, but we do not think that it is elementary in the mathematical sense, where what is most knowable and fundamental in itself is also most known to us. This would mean that all other types of classes are mere superstructures of what is only *unum coacervatione.* This was Democritus' opinion. (Aristotle, *VII Metaph.,* c. 13, 1039a5). As St. Thomas explains, it will affect the whole theory of numbers. «Et secundum hunc modum Democritus recte dicit, quod impossibile est unum fieri ex duobus, et ex uno fieri duo. Est enim intelligendum, quod duo in actu existentia, nunquam faciunt unum. Sed ipse non distinguens inter potentiam et actum, posuit magnitudines indivisibiles esse substantias. Voluit enim, quod sicut in eo quod est unum, non sunt multa in actu, ita nec in potentia. Et sic quælibet magnitudo est indivisibilis. Vel aliter. Recte, inquam dixit Democritus, supposita sua positione, qua ponebat magnitudines indivisibiles esse etiam rerum substantias, et sic esse semper in actu, et ita ex eis non fieri unum. Et sicut est in magnitudinibus, ita est in numero, si numerus est compositio unitatum, sicut a quibusdam dicitur. Oportet enim quod vel dualitas non sit unum quid, sive quicumque alius numerus; sive quod unitas non sit actu in ea. Et sic dualitas non erunt duæ unitates, sed aliquid ex duabus unitatibus compositum. Aliter numerus non esset unum per se et vere, sed per accidens, sicut quæ coacervantur.» (*Ibid.,* ect. 13, n. 1589).

There is a sense in which all classes are exclusive and closed, membership being restricted to those things which have the characteristic of the class. The classes associated with the type of variable we are discussing are confined to those things which have a form (a) in common. Any polygon belongs to the polygon class, because it is a many-sided figure. In this respect, the class of polygons is closed to any non-polygon. Since this exclusion follows from the identity of the common definition, even if there were an actually infinite multitude of members, it would still be closed in this sense.

This restriction of a class, which is due to the identity of the form (a), does not imply a limitation of the multiplicity (a). The class constituted by the proximate species of triangle, that is: equilateral, isosceles and scalene, is closed within and by itself both as to its form and to its matter (a). The class of isosceles, however, or the class of scalene triangles, allows ever new members without end; their possible varieties are indefinite within the confines determined by the common definition[1]. We may therefore say of such a class that it cannot *actually* contain all the members it *can* contain. Compared to a class which can have only a given number of terms, this kind of class remains forever open. However, because the term *open* may be used for divergent series, and *closed* for convergent series, we shall henceforth use the term *infinite*.

This property of an infinite class may seem contradictory, since «all the members of a class» means «all the members contained within the class». The contradiction arises only when we overlook the term «actually». An infinite class is a class which *can* have ever more members, that is, there is no end to the multitude or the variety of members it *can* actually contain. We may therefore define an infinite class: a class of which there may always be yet another member. This is a case of infinity as Aristotle defined it: that of which there is always something outside[2]. If a class is infinite in this sense, *all* its possible members can never *be*. On the other hand, the multitude of an actually infinite class would be beyond the reach of any possible number[3]. If there were such a class, it would be entirely determined and exclusive both as to its form (a) and as to its matter (a). It would be both infinite and perfect in multitude. We could apply to it the definition of a perfect whole: «that of which there is nothing outside»[4].

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1. We shall call «different by definition» anything which is more than numerically different; in Aristotelian terms: any difference which cannot be accounted for by intelligible matter. Thus, the circle *a* differs from the circle *b* by definition, and not by designation alone, if its radius is greater or smaller. (Cf. St. Thomas, *In VII Metaph.*, lect. 10).

2. *III Physic.*, c. 6, 207a.

3. «...Qui diceret aliquam multitudinem esse infinitam, non diceret eam esse numerum, vel numerum habere. Addit enim numerus super multitudinem rationem mensurationis: est enim numerus multitudo mensurata per unum, ut dicitur *X Metaphys.* Et propter hoc numerus ponitur species quantitatis discretæ, non autem multitudo; sed est de transcendentibus.» St. Thomas, *In III Physic.*, lect. 8, n. 4.

4. Aristotle, *loc. cit.*

If we did not make this distinction between the respect in which every class is an exclusive and perfect whole, and that in which some classes are essentially imperfect wholes, we might refer the potential infinity of a class merely to our inability to reach the class as a perfect totality, both as to form and as to matter. In other words, we might suppose that any infinite class is fundamentally a perfect class, i.e., that in itself, apart from our consideration, any infinite class has an actually infinite multitude of members, but that we, for some reason or other, cannot actually exhaust the actually infinite multitude. This would lead us into a maze of contradictions which some authors have heartily accepted. It is said, for instance, that the class of even numbers, or the class of odd numbers, has as many members as the class of integers of which it is a part. This is either a careless statement of the fact that both classes have the same *power* (which might be an unavowed way of reintroducing the concept of potency), or, at best, a good inference from a false notion of infinity. It is difficult to see how this contradiction could be avoided without the distinction of act and potency. But then, of course, Aristotle must be avoided, even at the cost of accepting a contradiction as a marvellous achievement. If the failure to make this distinction were logically carried through, it would be difficult to see how one could avoid destroying the very foundations of the method of limits[1].

5. The class of our variable ordered to a limit must be an infinite class in the sense we have defined. (We shall henceforth always take infinite for the potential infinite.) Infinity is essential to the variable ordered to the type of limit with which we are concerned. Let us now consider the type of order such a variable must comprise.

We may distinguish two kinds of order: accidental order, and formal or *per se* order. The order constituted by individuals of the same species, considered in their pure homogeneity, is merely accidental, such as 2, 2, 2,... The order of any pure aggregate where the formal differences of the members are not under consideration, is accidental, such as 3, 8, 13,... or toothbrush, star, pork-chop,... The order is formal when the formal differences of the members have, not just a common principle, but a common principle of their very differences, such as 1, 2, 3,... 1, 2, 4,...

Triangle is a variable whose proximate matter is equilateral, isosceles and scalene. While the order of these species is formal, it does not meet the requirements of the variable ordered to a limit. The order on the part of our variable must be formal and infinite. The infinity must be one of formal order. Only because of this infinite structural variation of its matter (a), may the variable itself be said to get closer to the form of the limit without end.

1. It may be pointed out that the difference between Aristotle's conception of infinity and the one we have just alluded to, is the difference between a dynamic notion of infinity and a static one. Potential infinity has meaning only with reference to an operation which may be carried on indefinitely. In the other conception, the operation is carried on in pursuit of what is already given.

We can now see the importance of the distinction we made between the symbolic form (c) and the intrinsic form (a). If we had not made this distinction, we could not be clear about the meaning of «the variable tends toward...» No tendency can be attributed to the form (a). We cannot say, absolutely, that polygon tends toward circle. The polygon that tends toward circle is the symbolic whole, that is, the polygon as a variable. We may then say that its identity tends toward otherness, its otherness toward identity, etc. However, the unity (a), the identity (a), the form (a), and the fixity or invariability (a) show that the tendency can never get beyond the stage of «tending toward...»

II. THE LIMIT OF THE VARIABLE

6. As regards the limit, or fixed term, or constant, much has already been said by implication. We may now state more explicitly that the limit may be compared to the variable in three ways: to the form (a) of the variable, to the matter (a), or to the symbolic form (c). The same would hold for the other properties of the variable. In the first respect, variable and limit are absolutely and irreducibly heterogeneous; they differ by definition. No variable can have a limit by virtue of its form (a) as such.

From this we may immediately infer that if a variable could actually reach its limit, the form (a) of the variable and the form of the limit would be both *formally different and formally identical as to their proper definition.* This is an instance of contradiction. It should be noted, however, that this contradiction at the limit is an essential condition that terms be related as variable and limit.

When we compare the limit to the matter (a) of the variable, we encounter a unity entirely foreign to the first comparison; for, any subsequent value of the variable is less different from the limit than any preceding value. Hexagon, of course, is closer to circle than triangle. As the series goes on, the difference decreases; and, since the series is infinite, there is no limit to the decrease of difference. Unless we arrest the movement of the series (and we will show later on that movement is essential to our variable), the difference is ever smaller and never definite. In this very determined sense we may say that the symbolic difference between the variable and its limit is indefinable, just as the position of an object in movement, as it is in movement, is indeterminable. On this level, there is no measurable difference that *is*[1]; and because the difference of the variable as tending toward the limit can never *be,* we may say that the symbolic otherness (d) of the variable tends toward identity with the limit. But just as it would be contradictory to say that the difference of the variable as such is a definite measurable difference that can *be,* it would be contradictory to say that this identity is definitely possible. The dynamic otherness of the

1. We here take the term *is* as restricted to what is only in act and to what is only in potency. In this sense, what is in movement, as it is in movement, *is* not. If the *est* in «Quodlibet est vel non est» were taken in this sense, becoming would be contradictory. Even contemporary Thomists have understood this axiom in a Parmenidean sense.

variable is always bounded by the indivisible identity of the form (a), as well as by the very inexhaustibility of the series which allows an ever decreasing difference. In other words, the very infinity which makes it possible to approach the limit without end and to overcome any possible given difference, is at the same time a reason why it is impossible to exhaust all possible difference. The undefinable difference will always *be*. We can also see that to say we cannot reach the limit because of the limitation of our mind and means of operation, is to deny the very infinity we would claim otherwise to exhaust[1].

III. THE TENDENCY TOWARD THE LIMIT

7. We cannot say that a circle is absolutely the limit of a regular polygon. Furthermore, we cannot say that it is the limit either of any given polygon, no matter how great or of any given series of polygons, no matter how great the given series may be. If there were a «greatest possible» polygon, a circle would in no sense be the limit of the polygon. That is why we say that a circle is the limit of «a regular inscribed polygon whose sides increase in number without end». The term «increase» must be taken in its dynamic sense. It is only with respect to the moving series that the limit is properly limit, and not with respect to some given or possibly given result, even though a pre-assigned positive number is always involved. A variable has a limit because of the possibility of an ever closer value; «ever closer» being taken dynamically, the infinity of the series has no meaning, when separated from the idea of sustained process.

The expressions «ad infinitum» and «in infinitum» signify not just an order, but a process. To say that a polygon and a circle meet at infinity, is quite misleading. If we thought we could conceive their meeting at infinity as possibly given, we would be denying the very infinity where they are supposed to meet—as if «at» infinity were a *locus* where they could meet. Such would appear to be the case if infinity could be exhausted, but an exhaustible infinity is a contradiction in terms, like a non-square square[2].

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1. «If the series has only a finite number of terms, we come at last in this way to the sum of the whole series of terms. But, if the series has an infinite number of terms, this process of successively forming the sums of the terms never terminates; and in this sense there is no such thing as the sum of an infinite series.

«But why is it important successively to add the terms of a series in this way? The answer is that we are here symbolizing the fundamental mental process of approximation. This is a process which has significance far beyond the regions of mathematics. Our limited intellects cannot deal with complicated material all at once, and our method of arrangement is that of approximation. The statesman in framing his speech puts the dominating issues first and lets the details fall naturally into their subordinate places. There is, of course, the converse artistic method of preparing the imagination by the presentation of subordinate or special details, and then gradually rising to a crisis. In either way the process is one of gradual summation of effects; and this is exactly what is done by the successive summation of the terms of a series.» A. N. Whitehead, *An introduction to Mathematics*, The Home University Library, 1931 pp. 197-8.

2. The illusion of a static infinity in this field may in some instances be due to a misinterpretation of those symbols which represent the infinity as a given «thing», whereas they should always be interpreted as standing for a process going on without end. But in most instances the illusion is associated with the following argument: of a given line we may take as many points as we wish, without end; but this is presumably possible because there *are* that many. The term *are* is taken univocally, and «that many» for a constant value.

Now, when we speak of the «possibility of an ever closer value», it should be understood that the only possible act is the fluent «act of getting ever closer». The act by which this possibility is defined is not like the actuality of a house, but rather like the act of that which is in potency as such: it is the possibility of getting ever closer, and not of actually reaching that which is approached. We must be equally careful, however, not to consider the act of the movement itself as the term of the movement. This would be contradictory, for the limit is the term of the movement; the movement itself is not that term. All movement is toward something other than itself, just as any relation is «esse ad»; and just as some relations are by nature such that they cannot «be in» that «toward» which they are, so some movement cannot actually reach the term «toward» which it is moving. We shall analyse this in a following paragraph and abide here by the mere indication of the distinction.

The definite act which the movement can reach is an act which is closer (than some other) to that toward which the movement is tending, while the limit is never the limit of any such act, but of the act of getting closer—which is the movement itself. Therefore the limit is the term of the movement itself. Whereas the term of movement in the ordinary sense must be defined by the possibility of actually reaching it, whether it shall be reached or not, the limit is the term of the movement «*qua* getting closer» in such a manner that if the getting closer ceased, the limit would no longer be a limit. The movement toward the limit cannot get beyond the stage of «being toward», although the «being toward» increases in actual value as the movement proceeds, and to this, again, there is no limit.

For this reason we may say that the variable getting ever closer to the limit, tends to enclose the limit as its own ultimate value. The converging series tends to bound itself by reaching beyond itself, i.e. by ever reaching beyond any value that is actually given within itself; the form (c) of the variable tends toward homogeneity, and therefore, toward formal identity with the limit. The variable tends somehow to disrupt, to break through, its own form (a), to negate itself and to assume the form of the limit. This should explain some of the more hyperbolic assertions of dialectical philosophies.

The «more» and the «less» of the formal order of the variable is not just quantitative. The increasing sums and diminishing fractions of the numerical series tending toward 2, the doubling and decreasing sides of the polygon, relate the variable to the formal structure of the limit. Some given value is not just greater than any preceding one; it is also more *like* to the limit. The *form* of the limit is the principle of comparison, a form which is other than the form (a) of the variable and which lies beyond any possible form (b) of the series. No possible value of a convergent series is «most like» to the limit. If, *per impossibile*, a «most like» value were possible, it would be identical with the limit. The polygon most like to a circle would be a circle; the sum of the numerical series $1+\frac{1}{2}+\frac{1}{4}\ldots$ «most like» to 2 would be 2.

The inseparable «more» and «less» characteristic of the variable ordered to a limit, pervade the series as a whole. Now, as we have seen, the «whole series» of an infinite series is essentially dynamic. The more and the less of the variable as such, are essentially dynamic, fluid. Because of this we may say that the variable tends to generate the limit from the *Wechselwirkung* of the great and small that are its matter; yet the great and small, the more and the less, are indestructible.

8. The limit being a constant, we cannot apply to it the distinctions we made in the variable. However, speaking elliptically, we might say that, in the limit, all the elements that make up the variable are identified. It is like a class with one member only. The limit may be variously related to the variable. It is primarily related to the variable as a whole, while we may also compare it more distinctly to the matter of the variable. We then call it the limiting value of the series; it is *as if* limit were the ultimate value of the variable, *as if* it belonged to the matter of the variable. If, *per impossibile,* the series could reach its limit, the limit would be the ultimate value *within* the series, that is, intrinsically part of the series and subject of the same proximate common form (a). The limit of a polygon would be both a circle and a species of polygon, that is, both a circle and a non-circle, a one-sided and a many-sided figure, an unbroken broken line, etc. On the other hand, no limit would be a limit if it did not have and retain its proper definition as other than that of the variable. The circle which we define: «a one-sided plane figure whose......», and the circle defined: «a limit of a regular inscribed polygon......», are the same circle. Hence, to attain the limit would be identical with destroying it. If it were not essentially other than any possible value of the variable, it would not be the limit of the variable. We may add that the limit which would be reached, would not *have been* the limit of the variable.

When it is said that the movement of the variable toward the limit is a movement toward contradiction, it must be understood that the variable never reaches beyond the stage of «being toward». The contradiction that *would* be is only «at the limit», but «at the limit» cannot *be.* As we have mentioned before, the movement toward a limit may be compared to a relation of reason. If the relation of identity between Socrates and himself were *in* Socrates absolutely, the very identity we wish to express by means of the relation would be just the contrary of identity: Socrates by the very fact of being himself would be other than himself: to be the same person would be, for him, to be two other persons; to be other than himself would be impossible without being non-other than himself, and so on *ad infinitum.* Furthermore, if the relation of identity were something in Socrates, it too would have a relation of identity to itself, and this one in turn, and so on[1].

1. «...Identitas est unitas vel unio; aut ex eo quod illa quæ dicuntur idem, sunt plura secundum esse, at tamen dicuntur idem in quantum in alique uno conveniunt. Aut quia sunt unum secundum esse, sed intellectus utitur eo ut puribus ad hoc quod relationem intelligat. Nam non potest intelligi relatio nisi inter duo extrema. Sicut cum dicitur aliquid esse idem sibiipsi. Tunc enim intellectus utitur eo quod est unum secundum rem, ut duobus. Alias ejusdem ad seipsum relationem

IV. THE BECOMING OF THE LIMIT

9. If it is already difficult to conceive what (*quid rei*) movement is—for movement is neither determinately act nor determinately potency—, it should be even more difficult to conceive what limit is; for, besides being movement, it is also rest, and it seems that it should be both determinately. If it is held that the Aristotelian analysis of movement renders obscure what is so clear[1]—clear if we intended to «save the name» only—, we dread to think what should then be said of the present attempt toward an analysis of what limit is.

So far we have considered the movement toward a limit as a state of becoming on the part of the variable. This state of becoming is referred to when we say that the variable approaches structural identity with the limit. This is the variable's state of becoming the limit. By running through its values, or rather by acquiring ever other and new values, the variable tends to become the limit in its very otherness. This tendency must be viewed in the light of the limit toward which it converges, i.e. in the light of what differs from the variable by definition.

There is a sense, then, in which the limit itself has a state of becoming. The «house becoming» is the state of becoming of the house that is to be. But the limit's state of becoming is not a becoming of the limit that is to be. The becoming of the house and the becoming of the limit would be comparable only if, *per impossibile*, the limit could be attained. When speaking of the limit's state of becoming, we must keep this proviso in mind.

The variable and the limit have each a double state: an absolute state defined by their irreducible identity, and a state of becoming. The variable is always other than its limit; but it is also becoming the limit, that is, becoming identical with the limit. The «becoming identical,» though,

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designare non posset. Unde patet, quod si relatio semper requirit duo extrema, et in hujusmodi relationibus non sunt duo extrema secundum rem sed secundum intellectum solum, relatio identitatis non erit relatio realis, sed rationis tantum, secundum quod aliquid dicitur idem simpliciter. Secus est, quando aliqua duo dicuntur esse idem vel genere vel specie. Si enim identitatis relatio esset res aliqua præter illud quod dicitur idem, res etiam, quæ relatio est, cum sit idem sibi, pari ratione haberet aliam relationem, quæ sibi esset idem, et sic in his quæ sunt secundum intellectum nihil prohibet. Nam cum intellectus reflectatur super suum actum, intelligit se intelligere. Et hoc ipsum potest etiam intelligere, et sic in infinitum.» St. Thomas, *In V Metaph.*, lect. 11, n. 912. From a purely rational point of view, this building up of relations of identity is far from being vain. Not only does it introduce a purely rational idea of infinity, but it generates relations of similarity and of equality and their contraries, as well as the whole system of abstract numbers. In this respect, the relation of identity is much more fundamental than the half-way notions of *any, some, all*, which we can produce from it.

1. «At vero nonne videntur illi verba magica proferre, quæ vim habeant occultam supra captum humani ingenii, qui dicunt *motum*, rem unicuique notissimam, *esse actum entis in potentia, prout est in potentia?* quis enim intelligit haec verba? quis ignorat quid sit motus? et quis non fateatur illos nodum in scirpo quæsivisse? Dicendum est igitur, nullis unquam definitionibus ejusmodi res esse explicandas, ne loco simplicium compositas apprehendamus; sed illas tantum, ab aliis omnibus secretas, attente ab unoquoque et pro lumine ingenii sui esse intuendas.» *Regulæ ad directionem ingenii*, edit. Adam et Tannery, pp. 426-427.—Descartes easily achieves his famous and mystifying *clarté* by ignoring the distinction between *quid nominis* and *quid rei*. Cf. Cajetan, *In de ente et essentia*, Procemium, edit. Laurent, n. 8, p. 19.

can never reach identity. This shows that the very expression «becoming identical», or «becoming identity», has no more than a purely dialectical meaning. The very meaning breaks down as soon as we would identify it with the becoming of a house, say. «A polygon and a circle meet at infinity» has true meaning only when interpreted: they would meet if, *per impossibile*, «at infinity» could *be*. Hence, «the variable is becoming the limit» must really mean: if the term toward which the variable tends could be reached, the variable would have to be determinately identical with the very otherness that is the limit in its absolute state. Provided we make this proviso, we may now add that in unceasingly acquiring its values, i.e. in accomplishing itself, the variable is at the same time undoing itself and tending to vanish into otherness. This again shows us how the most paradoxical statements of dialectical philosophies may be interpreted in simple terms.

The same may be said of the limit: it has a state of absolute identity with itself, in which it is absolutely other than the variable; but it also has a state of becoming, a state of «coming from» the variable. In other words, the limit must be coming from the otherness that is the variable, *as if* it were precontained in that otherness. The variable—whose proper values are being more and more actualized, so that the variable itself is becoming more and more the self that it ever more can be— must at the same time be moving away from itself and becoming identical with what is otherness to it, viz. the limit. We regret the deflating implications of this open statement of dialectical paradoxes. It shows they are *only* dialectical. When understood in the manner we believe to be the only sensible one, they do not have the formidable and exclusive connotation they are purported to have.

The very absolute identity of the limit with itself may be considered in two ways, provided the limit is itself something definite and given. It may be considered absolutely, and it may be considered with respect to the becoming of which it is the term; for, the absolutely given term, such as 2, is also, as to that very absoluteness, the limit which the series 1, $1+\frac{1}{2}, 1+\frac{1}{2}+\frac{1}{4}, \ldots$ is tending to be. This absolute identity, taken relatively to the series is, obviously, distinguished from what we have called «becoming identity»; it is the term because of which the becoming is denominated identical. The «becoming identical» is predicated of that which would be identical with the absolute identity of the limit if, *per impossibile*, it could be reached. If it could be reached, then the variable's state of absolute identity and the limit's state of absolute identity would be identical.

Hence we might also say that the limit, in its state of becoming, is the limit becoming identical with itself, that is, with its own state of absolute identity. Consequently, since the variable's state of becoming the limit is also the limit's state of becoming itself, the state of becoming of the one and that of the other are the same state of becoming, just as the road from A to B is the same as the road from B to A.

This shows that neither the limit nor the variable as such can be identified with either one of their states to the exclusion of the other. If the limit, say, were identical with its state of becoming on the one hand, and on the other hand with its state of absolute identity, its state of becoming would be identical with its state of absolute identity.

This would raise a false problem, if the two states of the limit were comparable to Socrates' state of rest and his state of walking. However they are not comparable, since the contrary states of the limit must be taken simultaneously, as contraries in the same subject at the same time. Yet this is contradictory. Does not this mean that the very notion of limit is absurd? The apparent contradiction is avoided when we realize that the contraries in question are not represented as *being* simultaneously in the subject. They are represented as *tending* to be in the same subject. To suppress the tendency would destroy the notion of limit.

It is the «tendency» which gives the variable and its limit that peculiar unity. This unity envelopes the contraries only as principles of the tendency toward identity, an identity which is actually no more represented than reached. The tendency convolves the states of the variable and of the limit into one inseparable whole, as can be seen from the fact that the whole structure breaks down as soon as we drop out any state of either term.

V. THE ABSOLUTE GENERATION OF A LIMIT

10. The becoming of the limit may be called its process of generation[1]. We must be careful, however, to distinguish what we here call generation, from natural generation, not merely because the former can never be accomplished, but because the novelty of the limit coming to be, is radically different from the novelty of some concrete individual nature such as Socrates. The generation of the limit is the generation of the very abstract nature itself, i.e. of that of which there cannot be *per se* generation. The 2 that is becoming in the series $1+\frac{1}{2}+\frac{1}{4}\ldots$ is not just «some» 2, but «the» 2. Twoness itself is becoming. The novelty, then, concerns the very «what it is» of the abstract nature itself, and hence, its very knowability «secundum se» is coming to be.

In other words, we treat the «given» 2 as if it were irrational in so far as it is merely given. It is as if 2 could become rational, that is, fully intelligibile and wholly grasped, only by such a generation. The givenness of 2 in its absolute state is, with respect to its becoming, as a state of irra-

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1. The natural generation here used as a term of comparison is taken in the broad sense described *In IX Metaph.*, lect. 7, nn. 1853,1854:—" . . . Omne moveri præcedit motum esse propter divisionem motus. Oportet enim quod quacumque parte motus data, cum divisibilis sit, aliquam partem ejus accipi, quæ jam peracta est, dum pars motus data peragitur. Et ideo quidquid movetur, jam quantum ad aliquid motum est. Et eadem ratione quidquid fit, jam quantum ad aliquid motum est. Licet enim factio in substantia quantum ad introductionem formæ substantialis sit indivisibilis, tamen *si accipiatur alteratio præcedens cujus terminus est generatio, divisibilis est*, et totum potest dici factio.»

tionality becoming rational in the structural change of the variable. The very givenness of the absolute state of any limit is as a barrier to its complete rationalisation. It is as if the mind could take fuller possession of 2 only by trying to exhaust the infinite that separates it from 1 or from 3. Hence, to take full possession of any absolute nature along these lines would involve a contradiction.

If the very givenness of 2, say, involves irrationality, then, in trying to rationalize 2, we must try to rule out, as much as we possibly can, that very givenness. This attempt can be most fully performed when a limit is the common limit of a lower and an upper series, as in the case of the number 2, which is the upper limit of the series (a): $1, 1+\frac{1}{2}, 1+\frac{1}{2}+\frac{1}{4}, \ldots$, and the lower limit of the series (b): $3, 3-\frac{1}{2}, 3-\frac{1}{2}-\frac{1}{4}, \ldots$ When we move in one direction alone, we move toward 2 in its absolutely given state, and there it escapes pursuit as if it were forever receding into an open background. Its becoming rational lies forever before it, the rear remaining safe. As the higher limit of one series it escapes circumvention, and the generation remains unilateral, as it were; but the direction of the withdrawal is somehow arrested and overcome when, at the same time, we move toward 2 from the opposite direction. The same holds for the circle when we approach it through inscribed and circumscribed polygons.

This consideration introduces an entirely new idea. For, when a limit is both upper and lower, the given absolute state of the limit may be relegated to the background. In other words, the limit we are tending toward in the series (a) is not the limit in its already *given* state, but rather the limit that is *coming to be* in the series (b), and *vice versa.*

$$\xrightarrow{\qquad} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \xleftarrow{\qquad}$$

$$3, 3-\tfrac{1}{2}, 3-(\tfrac{1}{2}+\tfrac{1}{4}), 3-(\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}), .., S_n, .., 2, .., S_n, .., 1+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}, 1+\tfrac{1}{2}+\tfrac{1}{4}, 1+\tfrac{1}{2}, 1$$

$$\lim_{n \to \infty} S_n = 2$$

This is what we shall call the absolute generation or rationalisation of a limit. It is, essentially, an unending process.

To consider the limit from the viewpoint of its absolute generation implies a marked advantage but also entails a difficulty. When we say that the limit we are moving toward in the lower series is, not the given absolute state of that limit in its otherness lying wholly beyond that order, but the limit that is coming from the upper series, we seem to rule out one condition which we have so far asserted to be essential. That essential condition was that any limit must have an absolute state in which it is absolutely determined otherness, unattainable and irreducible, whether it has in itself some definite value or not.[1] Now the only otherness we

1. We are deliberately postponing consideration of the case of the square root of 2.

choose to consider is the one coming from the other direction. This difficulty, though, is only apparent. We still suppose that otherness in two ways. We first suppose as already established that a given limit is the upper limit of one series and the lower limit of another, the one independently of the other. We then bring the two together, and henceforth define absolute limit as the limit toward which one series is moving, not in its absoluteness alone (although we do suppose that) but in its absoluteness as coming from the other series. Therefore, we have not eliminated anything. We have merely added a new respect based on a comparison between the two series *qua* converging toward one another. Although in this greater complexity we have become more independent of the absoluteness of the limit, we still presuppose it. It is only the independence of the mode of approaching it that has increased. This absolute generation is particularly interesting because, as we shall see later on, it offers a vantage point from which absolute mobilism and absolute immobilism may be dialectically reconciled.

(*To be continued in the next issue*)

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