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John Hannah

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The Doctrines of Triangles: A History of Modern Trigonometry by Glen Van Brummelen

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Reviewed by
John Hannah*
University of Canterbury
john.hannah@canterbury.ac.nz

The Doctrine of Triangles is the second half of Glen Van Brummelen's history of trigonometry. The first part, *The Mathematics of the Heavens and the Earth* [Van Brummelen 2009] dealt with the early history of trigonometry, from "precursors" in ancient Egyptian and Babylonian sources (dating from the second millennium BC), through to the trigonometric tables of Georg Rheticus in the 16th century AD. The present volume begins with a brief account of the story so far and then continues the story through to the early 20th century. It will be helpful to review briefly that "story so far".

Trigonometry was born of a desire to understand the heavens, and the first known quantitative modeling of the motion of the heavenly objects was done by ancient Babylonians in the eighth century BC. However, these calculations didn't involve any geometric models and so, for Van Brummelen, the real story of trigonometry begins a bit later, with Greek astronomers using geometric models to predict events such as eclipses and first risings. Since these models used circular orbits, and since astronomical observations involved angular displacements, it was useful to be able to convert angles at the center of a circle into lengths of the chords determined by such angles, and vice versa. Hipparchus (second century BC) seems to have been the first person to construct a table of arcs (equivalent to angles) and corresponding chords, but the earliest surviving table comes from Ptolemy's *Almagest* in the second century AD.

Ptolemy also showed how his table was constructed. and his techniques set the pattern for building many chord (and, eventually, sine) tables over the

* John Hannah is an adjunct fellow (retired lecturer in mathematics) at the University of Canterbury, New Zealand. For many years he taught a senior undergraduate course on the history of mathematics. He is particularly interested in the history of algebra and in the role of diagrammatic reasoning.

next thousand years or more. By viewing chords of certain angles as sides of constructible regular polygons, Ptolemy first calculates a basic set of chords (namely, those for 90, 72, and 60 degrees). In this case, "constructible" probably means that the relevant lengths can be calculated using nothing more complicated than square roots, but that ends up being equivalent to the polygons' being constructible by ruler and compass techniques. After proving some general rules for chords of sums and differences of angles, and of bisected angles, Ptolemy can then calculate chord values for all angles that are multiples of 3 degrees. Finding the chord of 1 degree is a bit more difficult, as there is no general "construction" for trisecting angles. Instead, he first proves that if angles α and β satisfy $\alpha > \beta$ then $\text{chord}(\alpha)/\text{chord}(\beta) < \alpha/\beta$, and then uses the known chord values for 3⁄2 and 3⁄4 degrees to deduce upper and lower estimates which prove that the chord of 1 degree must be 1;2,50 (to two sexagesimal places). Armed with this value, he produces a table for chords of all angles from ½ degree to 180 degrees, in steps of ½ degree. He also shows how to use what we would call linear interpolation to calculate chords for angles given to the nearest minute of arc.

Mathematicians have tended to think of their subject as developing linearly, each new advance depending directly on the previous one. But Van Brummelen shows that the early development and transmission of trigonometry is surprisingly nonlinear. Intermittent moments of cultural contact, separated by long periods of apparent isolation, seem to have led to hundreds of years of relatively independent development of trigonometry in three separate cultures: Greek, Islamic, and Indian.

It seems that the first transmission (or possibly, an independent development) saw the appearance in India of a Babylonian version of astronomy, in the form of astronomical period relations, possibly in the fifth century BC. Around the third or fourth century AD, a version of Greek astronomy and chord tables seems to have reached India. Somewhat surprisingly, it was not Ptolemy's version but something cruder. Almost immediately, though, the chord tables were replaced by sine tables (roughly speaking, $\text{chord}(2\theta) = 2\sin(\theta)$), the advantage being that this simplifies many astronomical calculations. Perhaps because Indian astronomy and mathematics were part of a mostly oral culture, these sine tables are quite short, with a step size of 3.75 degrees (equivalent to a step size of 7.5 degrees in a chord table), and the gaps are sometimes filled in by second-order interpolation techniques which were also described briefly (in verse).

The birth of Islam in the seventh century AD brought a third culture into the mix. Situated geographically between the Greek and Indian spheres of influence, it soon reached out to both cultures. Within a century or two, Islamic scholars had access to up-to-date versions of both the Indian and Greek versions of astronomy and trigonometry and were developing their own synthesis. In the case of trigonometry, the convenience of sine tables won out over the chord tables, but they adopted Ptolemy's methods for calculating the table entries. They also made their own innovations. Work on the construction and use of sundials led to the development of shadow tables, which would later become tangent and cotangent tables. Trigonometry was also used to solve characteristically Islamic problems, such as determining the beginning of the month Ramadan and the direction of prayer toward Mecca. This last development meant that trigonometry was now the mathematics of the Earth as well as the heavens.

Islamic influence spread across North Africa and into Spain, and by the 12th century Latin scholars began to get access to some of the works of Islamic scholars. European Renaissance writers tended to see this as simply (part of) the return of classical culture to its original home, but in any case it sowed the seeds of trigonometry in yet another culture. Whether it was the same culture as the Greek one that originally gave birth to trigonometry is a bit doubtful, of course, but the net result (either way) was that trigonometry now thrived in three living cultures (Indian, Islamic, and "European"). However, distance was not conducive to easy communication, and friendly relations between these cultures were at best intermittent. So, to the lasting discomfort of linearly minded (tidy?) historians, trigonometry developed more or less independently in all three cultures. Thus, for example, neither al-Kāshī's 15th-century calculation of sine of 1 degree by using an iterative method to solve a cubic equation, nor Mādhava's late 14th-century discovery of the power series for sine and cosine, reached Europe in time to influence similar developments there.

In both volumes of his history, Van Brummelen deals with this complicated situation by covering each culture separately. Thus, by the start of *The Doctrine of Triangles*, he has already covered developments in India and in Islam; and he has brought the story in Europe up to the mid-1500s. In this second volume, he follows the European story through to the start of the 20th century and, almost as an aside, covers a fourth cultural strand of trigonometry that developed in China.

At this point I found it useful to recall my own introduction to trigonometry. It may conjure up similar memories for some of my older readers. I first met the subject as a 13 year old at high school in 1966. In our regular mathematics classes, we learned about geometry and algebra. Geometry was about angles, parallel lines, triangles, circles, chords, and tangents, all presented more or less in the style of Euclid. Algebra seemed to be about pretending that letters could be numbers. It took me ages to figure out what the teacher was doing when there was a minus sign in front of a set of brackets. We were also introduced to logarithm tables as an easy way to multiply numbers. (Slide rules came along a couple of years later, and mechanical calculators did not appear until our physics labs at university.) Even with logarithms, negative numbers caused me confusion when some logarithms had a negative whole number part (indicated by a bar over it) and a positive decimal part. Then, once a week, at the usual time for our mathematics class, a different teacher introduced us to trigonometry, which seemed to be about calculating sides and angles in right-angled triangles. I assumed this new teacher was more expert at the subject than our regular mathematics teacher, possibly because he also taught technical drawing to the less academic boys in the school. A key tool in trigonometry was another set of tables which listed angles and corresponding ratios called tangents ("opposite over adjacent"). No one ever told me that these two uses of the word tangent were related. Because of the context, all the angles were strictly between zero and 90 degrees. Because we did not meet functions or graphs until the following year, tangents came only as tables. Sines and cosines were not introduced until the following year.

As the curtain rises in 1550 in this second installment of the history of trigonometry, sine tables have been around for a 1000 years; but they list lengths (of half-chords in circles of various standard radii) rather than ratios, as we now think of them. Tangent and cotangent tables, in the form of shadow lengths corresponding to varying heights of gnomon, have been around since at least the time of al-Khwārizmī in the ninth century. However, tangents are not yet called tangents, even though Abū'l-Wafā had drawn the trigonometric tangent as a tangent line in his classic diagram of all six standard trigonometric quantities as long ago as the 10th century. Indeed trigonometry itself had not yet received its modern name. That did not happen until Pitiscus [36] coined a New Latin title for his *Trigonometriae* in 1595. But trigonometry in the guise of a doctrine of triangles was alive and well. On the other hand, algebra, logarithms, and even a general acceptance of negative numbers were still in the future, albeit the relatively near future. Although *The Doctrine of Triangles* is certainly history written from the point of view of the people of the time, the basic narrative thread is essentially the story of how sine and cosine (and tangent) developed from lengths, or ratios of lengths, to being functions acting on any real (or even complex) number. Not just that, they are differentiable functions that satisfy the wave equation and, through Fourier Series, they can also be used to solve the heat equation. It is very tempting to read this amazing story as a “royal road to us”.

The Doctrine of Triangles opens (chapter 1) with an account of what Van Brummelen calls the coming of age of European trigonometry. Once again, the story is far from linear, as competing presentations of trigonometry include different “functions”, picture these functions in different ways, and measure them against different “radii”. Presumably, history then selects the fittest for survival. For example, Regiomontanus in his major work, *De triangulis omnimodis* (published in 1533), pictures the sine, cosine, and versine as lengths in a circle of radius $R = 100{,}000$; but he omits any form of the tangent [4]. He did include the tangent in his *Tabula directionum* (published in 1490), but it is called “tabula fecunda”. On the other hand, Rheticus, in his *Canon doctrinae triangulorum* (published in 1543), lists all six functions: sine, cosine; secant, tangent; and cosecant, cotangent and pictures them as these pairs, coming from three typical right-angled triangles in which either the hypotenuse, or one of the other sides (respectively) is the standard “radius” $R = 10{,}000{,}000$. In this case, trigonometry has almost been freed from its circular origins [9]. The large values for the radius R seem to reflect a desire to work with whole numbers, rather than with decimal fractions. Using even larger R values allowed table makers to use smaller step sizes and to display more accurate values.

Although Cardano had shown Europeans how to solve cubic equations in 1545, all of these ever-finer tables were still built up using essentially the same technique as Ptolemy used when he calculated the sine of 1 degree. When Viète invented symbolic algebra in the late 1500s, he was able to offer an alternative route to finding the sine of not just 1 degree but also of 1 minute [27]. The key idea was to find recurrence relations for $\sin(n\theta)$ and $\cos(n\theta)$. Using his algebra he was then able to express $\sin(3\theta)$ as a cubic polynomial in powers of $\sin(\theta)$, and similarly for higher multiples of the angle. So the sine of 1 degree could be calculated by solving a cubic equation (as al-Kāshī had done almost 200 years earlier within the Islamic culture). Viète didn’t actually carry out the final calculation himself, but Briggs did in 1633.

In the late 1500s trigonometry in Europe crossed a different kind of cultural boundary. Trigonometry had begun as a servant of astronomy but now it began to be used in surveying and other “practical geometry” [46]. Typically practitioners of practical geometry solved problems by using similar triangles and the Pythagorean theorem. This would have been a bit like the cultural divide that existed 50 years ago between universities and polytechnics. I imagine my first trigonometry teacher, a technical drawing teacher, as a descendant of this tradition. Trigonometry's crossing of the divide is shown by Fincke (1583) and Clavius (1604), who wrote books showing how problems involving heights, distances, and lengths could be solved using tangents and other trigonometric functions.

Trigonometry clearly had potential uses in navigation too but, even if there had been no cultural divide, there would be serious practical issues to be solved first. Trigonometric techniques generally involved multiplying and dividing long numbers from the tables, and these calculations were slow and difficult to perform reliably. As schoolboys in the 1960s we faced the same problem with our trigonometric calculations. We were perhaps the last generation to experience the problem. In chapter 2, Van Brummelen describes how Napier, in 1614, solved the problem by inventing logarithm tables. Suddenly multiplication and division were more or less as easy as addition and subtraction. As with trigonometric tables, Napier's logarithms were all whole numbers ($10{,}000{,}000 \times \log(x)$, in our terms). Unfortunately, some logarithms were negative and some positive, which led to errors in calculations; and it was not long before Speidell (1627) hit on the idea of subtracting all Napier's logarithms from 100,000,000, thus making them all positive [73]. As I mentioned earlier, this idea persisted through to the log tables of my youth, although it was only on reading this book that I realized what had been happening! It seems to be clear that Napier was motivated by the trigonometric calculations used in astronomy, but we have seen that the need to simplify calculations extended to other "professions"; and it was not long before trigonometric and logarithm tables were published not just in Latin (the usual language of scholars) but also in the contemporary languages of Europe. In other concessions to practical needs, smaller sets of tables were produced, using fewer significant figures but more suited to use on sea voyages [75]. Incidentally, language barriers did not always cut European society the same way. Thus, Descartes originally published his *Géometrie* in French [95], but it did not become widely known until van Schooten translated it into Latin. The bonds that "united" Europe, and made it possible to conceive of a Eurocentric view of history, were sometimes quite tenuous.

Chapter 3 shows how trigonometric functions feature in the development of calculus in the 17th century. In the years leading up to the discovery of calculus, much effort was devoted to calculating the areas of regions enclosed by curves and to constructing tangents to curves. Here curves were generally defined by algebraic equations (following the invention of coordinate geometry by Fermat and Descartes) but sometimes they could be generated by a mechanical process. Thus Van Brummelen shows Roberval, in the 1620s or 1630s, calculating the area under a cycloid, a curve described by the motion of a point on the circumference of a rolling circle [113]. In this case a cosine curve appears as a result of a geometric transformation of the area sought, but Roberval also gives a direct construction of a sine graph and calculates the area under this graph. This seems to be the first acceptance that a trigonometric function could define a curve. By 1670, the curves associated with sine, cosine, tangent, and secant had all appeared in print [117]. Not surprisingly, given their origins, these curves were drawn just for angles in the first quadrant. The spur to considering larger angles seems to have come from looking at multiple angle formulas for the basic trigonometric functions. A formula for $\sin(n\theta)$ in terms of powers of $\sin(\theta)$ and $\cos(\theta)$, for example, could give rise to equations where one side of the equation made sense, but the other side involved angles larger than 90, or even 360, degrees. Cotes [148] gives graphs of the tangent and secant over several periods in 1722.

On another front, Newton's study of infinite series in the 1660s showed that trigonometric functions could also be thought of as algebraic if we extend algebraic operations to infinite series. Van Brummelen [129] shows how Newton derived an infinite series for arcsine. By extending to infinite series the known arithmetical algorithms for long division and finding square roots, he finds an infinite series representing an element of arc length on a circle. Integrating this element gives the desired series for arcsine. Inverting this series, that is, finding its inverse function as a series, then gives the now-standard infinite series for $\sin(x)$. Within a generation infinite series, and with them the standard trigonometric functions, became part of the routine toolbox of mathematics. In the 1740s, Euler used series to solve differential equations and found the exponential and trigonometric functions involved together in solutions to oscillation problems [161].

Chapter 4 is an interlude dealing with trigonometry in China. As Van Brummelen says [xv], there was no obvious place to put this chapter. One consequence of China's relative isolation and independence is that it has (until very recently) had little impact on the development of "European" mathematics. As a consequence, there is no obvious point in the history of trigonometry, as there was with Indian or Islamic culture, where "we" could talk about "their" contribution to "our" story. In those two cases, the entry points also offer "us" the opportunity to continue "their" story after the two stories diverged, before taking up the main thread of "our" story once again. I can imagine Chinese historians discussing Indian, Islamic, and European contributions to the Chinese story in almost the same words. Indian contributions to the Chinese history of trigonometry probably began with the arrival of Buddhism around the first century AD, but the first surviving document [186] is a translation of an astronomical treatise from the seventh century AD. In the section on eclipses, there is a typically Indian table of sines in steps of 3.75 degrees. However, because of the treatise's astrological content, the document was kept secret and seems not to have had any lasting impact.

Van Brummelen looks for signs of a native trigonometry in China and there are some promising leads. There was a different unit for measuring angles, corresponding to the distance the Sun travels along the ecliptic in one day. Of course, our degree may have started life that way too, but the Chinese "du" corresponds to one year being 365.25 days rather than 360. Van Brummelen gives an example [188], from the eighth century, of a table of shadows for various angles measured in "du". Some third-century (AD) Chinese geometry comes close enough to trigonometry to cause European scholars to argue about the definition of trigonometry: Liu Hui shows how to calculate the dimensions of a walled city using sightings from afar. Van Brummelen leaves the reader to decide for themselves. Much later, in the 13th century, Guo Shoujing is supposed to have calculated a variety of astronomical quantities by using a home grown version of the versed sine [199]. By this stage some aspects of Islamic culture, in particular astronomical instruments, have reached China; but Guo seems to have worked within an indigenous tradition that was independent of such foreign influences. His own writings are lost and Van Brummelen has to rely on how later commentators have reported and interpreted them.

In the late 1500s, Jesuit missionaries came to China, keen to share European knowledge of science and technology, and keen too to collaborate with Chinese intellectuals. One outcome, for example, was a Chinese edition of the first six books of Euclid's Elements [203]. The Jesuits' idea of "sharing" probably amounted to what we would call transmission, but the differing outlooks of the two cultures at this stage in history made that an unlikely outcome. Euclid's system of axioms, theorems, and proofs, for example, was out of step with Chinese interest in measurement and computation. For Van Brummelen, the result is "a complicated narrative of appropriation and naturalisation" rather than simple transmission [204]. But as we have seen, this is what each culture so far has done to trigonometry when it arrived on their shores. In the Chinese case this appropriation and naturalization lasted for almost 200 years. Thus, trigonometric tables [206] and logarithms [211] came to China, but there was some disagreement as to whether angles should be measured in “du” or degrees; and if it was to be in degrees, should the degrees be subdivided into minutes or into hundredths? Other developments seem to be more original, and Van Brummelen devotes the end of chapter 4 to Chinese work on infinite series, culminating in Li Shanlan’s derivation [239] of a series for arcsine in the mid 1800s. Then a second influx of “Western” learning, including some translations by Li himself, followed by the political disruption of the Opium Wars, led to China’s eventually being incorporated into the global mathematical enterprise.

Finally, Chapter 5 returns to the story of trigonometry in the European culture, tracing developments after the time of Euler. Euler’s work had effectively incorporated trigonometry into the general mathematical toolbox that was available for solving other mathematical problems, as well as “real-life” problems from outside mathematics. Van Brummelen gives examples of trigonometry’s being used by Legendre to solve large-scale surveying problems where the Earth’s curvature needs to be taken into account [262] and of its being used by Fourier to solve the heat equation [285]. On the other hand, there was also progress within trigonometry. For example, in another reminder of life before computers, the search continued [250] for logarithm-friendly identities, that is, identities that used only multiplication, division, and square roots (as opposed to identities which combined these operations with addition and subtraction). For me, in a pleasing endnote to my experience of tables, slide rules, and calculators, Van Brummelen describes [280] an early (1950s) computer algorithm for sine and cosine. All my 13-year-old grandchildren need to do now is push buttons on a calculator.

Van Brummelen discusses many other topics. In particular, I have dismally failed to do justice to his coverage of spherical trigonometry. As usual, he is very easy to read, and there are lots of helpful diagrams, especially for the spherical trigonometry. As in the first volume of his history, the story is deeply enriched by extracts from contemporary texts, given first in fairly literal English translations, often accompanied by the original diagrams, and then explained in modern terms. So mathematical readers (and, I hope, their students) can experience a little of what trigonometry was actually like at each stage in its history.

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