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Résumé de l'article

Cet article est une introduction aux études de rebalancement dynamique en présentant les différentes méthodes déjà élaborées. Deux grandes voies ont été utilisées pour le rebalancement avec coûts de transaction : Optimisation locale et Optimisation globale. La première approche essaye de fixer le risque ou la période de rebalancement comme variable exogène. La deuxième approche propose de trouver un élément d'optimalité sous l'hypothèse de la maximisation de la fonction d'utilité sur toute la période de rebalancement. L'article est divisé en trois parties : une introduction au sujet du rebalancement dynamique, une revue des différentes méthodes élaborées et finalement quelques notes de conclusion sont proposées.

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DYNAMIC HEDGING UNDER TRANSACTION COSTS : A LITERATURE REVIEW

by Maher Yaghi

ABSTRACT

This article presents an introduction to dynamic hedging with a description of the different methods already used to implement a discrete hedging program. Two very distinct approaches have been taken in the pursuit of dynamic hedging under transaction costs : Local in time and Global in time. In the first approach, the hedge timing strategy is fixed exogenously or the risk taken is fixed exogenously. The second approach proposes to achieve an element of optimality under the utility maximization approach. In the first section of the article, we will discuss the background into dynamic hedging. The second section will elaborate on the different methods that have been proposed so far and in the final section some concluding remarks are discussed.

RÉSUMÉ

Cet article est une introduction aux études de rebalancement dynamique en présentant les différentes méthodes déjà élaborées. Deux grandes voies ont été utilisées pour le rebalancement avec coûts de transaction : Optimisation locale et Optimisation globale. La première approche essaye de fixer le risque ou la période de rebalancement comme variable exogène. La deuxième approche propose de trouver un élément d'optimalité sous l'hypothèse de la maximisation de la fonction d'utilité sur toute la période de rebalancement. L'article est divisé en trois parties : une introduction au sujet du rebalancement dynamique, une revue des différentes méthodes élaborées et finalement quelques notes de conclusion sont proposées.

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INTRODUCTION

Options are a part of the wide family of derivative securities, which count the futures, swaps, and forwards, etc. Compared to linear products such as futures or forward, options possess some special characteristics making them very flexible and can be used in a variety of ways. All we need to price an option is the existence of an underlying security and some parameters including risk-free rate, volatility, time to expiration, and a payoff function.

The seminal work done by Black & Scholes (B&S) in 1973 was the beginning of today's contingent claims analysis. In their paper, the authors created a risk-free hedge, which consists of a long position in a stock and a short position in an option. They go on demonstrating how this portfolio should earn the risk free rate thus opening the possibility in pricing stock options without the need for the underlying's growth rate or the discount rate of the option.

Option pricing and portfolio hedging are widely used in today's financial market in taking speculative positions, reducing risk exposure, capital budgeting and risk-free arbitrage to name a few. Options can exist in many shapes and forms, ranging from the basic calls and puts on stocks to the very colourful interest-rate Asian options or to real options used in insurance, capital budgeting or executive compensations. Much work has been done so far on the pricing and the use of these derivatives. This article will try to sort through the work already done in maintaining a risk-free portfolio. As mentioned before, maintaining this risk-free hedge is crucial because it is the foundation of all option pricing. Thus this article could shed some light on the possible existence of a risk-free hedge in a not so perfect world and lead the way to the implementation of a trading program that would be used in portfolio management, hedge funds and day trading to name a few.

BASIC MODEL OF OPTION PRICING

In the world of options, two domains have progressed to answer two different questions. The first is what should be hedged; this product could be called *risk management design*. The second question would be to answer the question of how to hedge; this is the *hedging technology*. In its construction, hedging technology is mathematically dependent and could take many different complex routes thus allowing for different interpretations. The results of these routes came to be as opposite as to accept or to refute even the possibility of is discrete hedging possible ?

Before Black & Scholes (B&S), the Boness (1964) and Sprenkle (1964) models depended on the underlying's growth rate and a discount factor for the option. In 1973, the B&S model was introduced and removed all estimates that integrate measurement and interpretation errors. By constructing a perfect hedge, they were able to use Itô's lemma and produce the partial derivative formula (PDF) followed by the option price:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

Or equivalently: $\Theta + \frac{1}{2}\sigma^2 S^2 \Gamma + rS\Delta - rV = 0$

where V is the option price, S is the underlying's price, σ is the underlying's volatility and r is the risk-free rate. We then find the first derivative to time ("Θ" theta), the second derivative to the underlying ("Γ" gamma), the first derivative to the underlying ("Δ" delta). The above equation is a linear parabolic partial derivative equation. Linear meaning that if there are two solutions, their sum is also a solution and parabolic meaning that they relate to a diffusion process. The notation table is the appendix.

The major assumptions of this derivation are :

1) The underlying follows a lognormal random walk
2) The risk free rate is a known function of time
3) No dividend on the underlying
4) Delta hedging is done continuously
5) No transaction costs on the underlying
6) No arbitrage opportunities.

In this article we will be interested in relaxing the fourth and the fifth assumptions. We should point out that the fifth assumption relates to the underlying, not the option since the B&S hedge consists of a long and varying quantity position in common stock and a short position in call option. This hedge will require a net investment that will earn the risk-free rate and could be called an investment hedge. This is in contrast to a borrowing hedge that consists in a long call position and a short stock position providing a supply of fund that will cost the risk-free rate. Later, we will be relaxing these assumptions to find that in certain situations, if transaction costs are very low, arbitrage could conceivably create society's lowest cost financial intermediary : a financial black hole. From this point forward, the discussion will be built around an investment hedge but the equivalent is also possible.

It also should be pointed out that the delta has a significant role in a hedge since it gives the quantity needed of the underlying to have a complete hedge. For plain puts and calls, delta changes from 0 to 1 for a call and from –1 to 0 for a put. Geometrically, delta can be expressed as the slope of the line drawn tangent to the option's theoretical price curve at the point equal to the underlying stock's value. Individual call options unless very deeply in the money or moderately in the money but very near to expiration, carry deltas less than one. The problem with deltas is not how to calculate them nor how to create them, but rather how to cope with their constant change. The gamma is an indication of how stable the hedge will be in case of small changes in the underlying. It is the sensitivity of delta. The delta of an option is in constant flux and gamma gives an indication on the direction of its movement. Finally, theta is the measure of the speed with which a given option loses time value.

Now that we covered most of the model's background, let us take a moment to review the steps of portfolio hedging. First, after deciding the portfolio of stocks to be hedged, a quantity of options on the underlying is shorted to create a risk free hedge. To keep the hedge risk-free, the underlying portion should continually be rebalanced and the quantity needed is calculated through a measure related to the dollar-delta of the option. The return of the portfolio is a combination of the return on the underlying and the option portions. If these actions were repeated until expiration time, the portfolio would earn the risk-free rate with daily returns uncorrelated with the underlying. The reason for this is that in Ito's lemma derivation, in continuous time, all but the first three derivations are kept and these derivations do not exhibit any systematic risk. This theory allows the portfolio to be priced in a way to have the risk-free rate of return.

The different proposed techniques have as final motive to create intervals that give the manager of the hedge the signals that he or she needs to keep the hedge in balance. Thus, by controlling for transaction costs and out of balance risk, the model will evaluate the best course of action. The signal should be clear and precise to automate as much as possible the rehedging process. At each point, the model should also give a clear signal on how much the hedge is out of balance and the risk in dollars that the manager is exposed to.

Relaxing hypothesis number 4: Authors verified that even in discrete time, with no transaction costs, the return of an option hedge is a function of the square of the change in the underlying and for short rebalancing it still exhibited little or no systematic risk. The Law of Large Numbers for Martingales states that the sum of the changes in the difference between the change in value of the portfolio and the call option will tend toward zero. Thus, the return on an individual hedge portfolio is uncorrelated with the return on the market, but the return on the hedge is not independent of the return on the market. Boyle and Emanuel (1980) put it simply that the return is the product of three components:

1) A deterministic function of the underlying evaluated when the hedge is constructed
2) A random variable drawn from a chi-squared distribution with one degree of freedom with mean of zero (because of the underlying's leptokurtic return distribution)
3) The time interval between adjustments

This derivation allowed later on for the construction of rebalancing techniques that used intervals of one day to one week (Whaley 1982) and even to two months (Leland 1985). These experiments showed that under *controlled volatility*, the dynamic hedge strategy leads to ex-post portfolio returns that approximate the ex-ante risk free return's theoretical value.

Any derivation should make sure the following possibility is accounted for : B&S were able to derive their formula on the basis that path-independent strategies are option replicating strategies, but what if the hedge has the potential for requiring to be fully invested. This means that the portfolio is completely composed of the underlying. This is a real possibility when we are close to maturity and the option is very out of money. In this case, the portfolio will exhibit substantial systematic risk. Thus, a dynamic hedge strategy that has the potential to be fully invested or fully cashed out is path dependent.

Another potential for error is the whipsaw effect that happens when the underlying is close to the strike price; in this region, the delta could greatly fluctuate thus any out of balanced portfolio could exhibit extreme systematic risks. In this case, a gamma neutral strategy could make the difference. Thus, any hedging should include a correction for the delta and the gamma, when needed, and to protect from extreme movements when the underlying is near the strike price. This will minimise movements in the portfolio since a gamma and delta neutral position has a greater probability of staying in the boundaries of no transaction than a solely delta neutral position.

In a first step, we will suppose that the distribution is lognormal allowing for the use of the usual normal or t-statistic values. Nevertheless, in reality, the underlying follows more a leptokurtic distribution thus the t-statistic will be biased. For a small sample size, the normal t-statistic will be biased downwards, this is derived from a closely bunched negative returns and the variance estimate tends therefore to be low. This leads to a large negative t-statistic. On the other hand, a positive mean return is caused by a predominance of positive widely dispersed daily returns making the t-statistic appear insignificant because of the resulting high variance estimate. We know that the law of large numbers will tend to favour the use of the standard t-statistic but trading steps during a certain period could be small thus correcting the t-statistic could make the difference. This corrected t-statistic will have a shortened lower tail and we expect it to help to take into account the frequent extreme outliers when there are few trading intervals and the option is initially at the money.

Relaxing hypothesis number 5 : In theory, since borrowing rates are higher than lending rates, the borrowing hedge option will be greater than an investment hedge's option price. In the case of transaction costs, in an investment hedge, just raising the option price can recapture these costs incurred. For a borrowing hedge, transaction, and market impact costs caused by large transactions, will push the option's price lower. If cost spreads are the right sizes, the investment and the borrowing hedges will be equal. If the cost spread is low relative to the lending and borrowing rate spread, the investment hedge option could be lower than a borrowing hedge option. Thus if an option can be replicated to be situated between these two boundaries, it will yield a lending rate higher than the risk free rate or a borrowing rate lower than the market's borrowing rate. This is what was previously referred to as a financial black hole. Risk free arbitrage can then be used to generate alpha (extra return).

We argue that modelling any discrete hedging without correcting for transaction costs will lead in many cases to accept the examined technique but when comes the time to include these costs, the model becomes biased. Much work has been done so far on this subject : Boyle & Vorst (1992) and Leland (1985) are good examples. As mentioned earlier, option replication is by construction a path independent strategy but not when there are transaction costs. This is because the final return will depend on the cost incurred that itself depends on the path taken by the underlying. Let us take for example a position on an underlying that is very close to the strike price. As mentioned earlier this possibility will result in a very high whipsaw effect inducing much rebalancing thus high rebalancing costs. On the other hand, an option very much in or out of the money will require less rebalancing thus lower transaction costs.

Any replicating strategy should respect some general guidelines. First, transaction costs should remain bounded and second, the return including transaction costs should be uncorrelated with the market as much as possible. Let us remember that we need two figures, a transaction cost on the option and a second for the underlying. In the real world however, rebalancing a hedge position will take a very different strategy. Many traders have seats on one of the option or the stock exchanges thus allowing them lower costs on trades. These traders will prefer hedging an option position with other options and not stocks to save on the higher costs.

LITERATURE REVIEW ON TRANSACTION COST MODELS

Two very distinct approaches have been taken in the pursuit of dynamic hedging under transaction costs : Local in time and Global in time. In the first approach, *the hedge timing strategy is fixed exogenously* : Leland (1985), Boyle & Vorst (BV) (1992), Hoggard, Whalley & Wilmott (HWW) (1994), Avellaneda & Paras (AP) (1994), Toft (1996), *or the risk taken is fixed exogenously :* Whalley & Wilmott (WW) (1993) and Henrotte (1993). The second approach proposes to achieve an element of optimality under the utility maximization approach Hodges & Neuberger (1998), Davis, Panas & Zariphopoulou (DPZ) (1993) and Whalley & Wilmott (1994, 1997).

In the following section, we will describe the two approaches and the different articles pertaining to them and new approaches coming in the horizon.

Local in Time

The Boyle & Emanuel model (1980)

The aim of the paper was to identify the component of the return of a discreetly balanced hedge portfolio in the presence of no transaction costs and to analyze the distribution of the returns on the portfolio. At time t a hedge portfolio is constructed by purchasing one call option at price C and selling $\partial C/\partial S$ units of stocks short so that the initial investment is $C - SN(d_1)$. At time $t + \Delta t$, the value of the portfolio is : $C + \Delta C - (S + \Delta S)\, N(d_1)$.

Hence, we obtain an expression for the hedge return in (1) and (2) if ϕ is a random variable drawn from a normal distribution with zero mean and unit variance:

$$HR = \Delta C - \Delta S N(d_1) + r\Delta t X \exp(-rt^*) N(d_2) + O(\Delta t^2) \quad (1)$$

$$= \left[C_t + \tfrac{1}{2} C_{SS}\sigma^2 S^2 u^2 + rX\exp(-rt^*) N(d_2)\right]\Delta t + O\left(\Delta t^{2/3}\right) \quad (2)$$

where

$$C_t = -X\exp(-rt^*)\left[\frac{Z(d_2)\sigma}{2\sqrt{t^*}} + r N(d_2)\right] \textit{ and}$$

$$C_{SS} = \frac{Z(d_1)}{S\sigma\sqrt{t^*}} \cdot$$

Substituting the values of C_t and C_{SS} in (2) and neglecting higher order of Δt we obtain

$$HR = \lambda\, y\Delta t \quad (3)$$

where

$$\lambda = \frac{\sigma S}{2\sqrt{t^*}} Z(d_1) \textit{ and } y = \phi^2 - 1.$$

HR is negative only if $|\phi| < 1$. Since ϕ is drawn from a unit normal distribution, HR will be negative 68% of the time and positive in case of large movements of the underlying. We can see from (3), that the hedge return other than being uncorrelated with the market can be decomposed into three components :

i. A deterministic function of the underlying evaluated when the hedge is constructed given by :

$$\lambda = \frac{\sigma S}{2\sqrt{T}} Z(d_1) = \tfrac{1}{2}\sigma^2 S^2 C_{SS} .$$

If the number of options held is kept constant, the return will be heteroscedastic. Adjusting the trading interval Δt inversely with λ could reduce this phenomenon.

ii. A random variable drawn from normal distribution : $\phi^2 - 1$. The authors point out the fact that in reality, this stochastic variable is more skewed than normal and thus any inference on hedge return should be adjusted to correct for homoscedasticity.

iii. The time interval between adjustments Δt

This model gave a very intuitive view of the expected return over a hedge but did not tackle transaction costs. However, the authors tested many different t-statistics (t-stat) adjustment techniques to assess their significance. They examined the biases that can arise from the skewness of the return distribution in the calculation of t-stat for confidence interval construction. Assuming λ constant, the estimation bias will exactly correspond to that obtained from a chi-squared distribution with one degree of freedom.

- The first approach is to use raw t-stat tables from normal distribution. It was found that this method failed to pick up fat tail distributions.
- Another approach to solving this is by constructing frequency tables of the estimated t-stat. These tables could be used for confidence construction like regular t-stat. The authors found that this approach worked well for in sample numbers but failed in out of sample calculations.
- The adjusted Johnson (1978) statistic that uses sample mean, variance, and skewness.

$$Adj.\,t = \frac{(\bar{x} - \mu) + (\mu_3 / 6\sigma^2 N) + (\mu_3 / 3\sigma^4)(\bar{x} - \mu)^2}{\sqrt{(s^2 / N)}} .$$

It was not found useful because it does not correct for the presence of heteroscedasticity present in hedge returns.

- A homoscedastic t-stat : this method did not help since it is not distributed like a chi-square.
- A doubly corrected t-stat : It was constructed by a combination of Johnson's statistic and a Homoscedastic one. This last measure proved resilient and gave good adjustments to fat tail in case of out, at or in-the-money options.

Thus, Boyle and Emanuel tried to identify the component of the return of a discreetly balanced hedge portfolio but did not include the transaction cost component.

The Leland model (1985)

This was the first complete article on the introduction of transaction costs into the dynamic hedging world. The paper proposes an alternative replicating strategy depending upon the level of transactions costs and upon the revision interval and the option to be replicated. The model leads to note that:

i. Transaction costs remain bounded, as the revision period becomes short.

ii. The strategy replicates the option return inclusive of transaction costs, with an error uncorrelated with the market and approaches zero, as the revision period becomes short.

iii. An analytical formula to calculate the expected transaction costs, and bounds on option prices.

The model assumes that the underlying follows a logarithmic diffusion process. The same portfolio as in the Boyle & Emanuel model is constructed and held for the length of Δt (the revision interval). In case of no transaction costs but discrete hedging, the return on the portfolio ΔH will be as in Boyle & Emanuel :

$$E[HR] = \tfrac{1}{2} C_{ss} S^2 \left[\sigma^2 \Delta t - \left(\frac{\Delta S}{S} \right)^2 \right] = 0.$$

The author shows that even with revision periods lasting 8 weeks, the expected return will be close to zero. In addition, halving the revision period reduces the standard deviation of the hedge return by a factor of exactly $1/\sqrt{2}$.

In the second derivation, Leland shows that by introducing a proportional transaction cost to the underlying's price, a closed form solution can be obtained by adjusting the volatility in the B&S model for a long (short) call.

Let k represent a round trip cost measured in fraction of the volume of transaction:

$$\hat{\sigma}^2\left(\sigma^2, k, \Delta t\right) = \sigma^2\left[1 \pm k\, E\left|\frac{\Delta S}{S}\right| \Big/ \sigma^2 \Delta t\right]$$

$$= \sigma^2\left[1 \pm \sqrt{(2/\pi)}\, k / \sigma\sqrt{\Delta t}\right].$$

Integrating this into the B&S portfolio : $\Delta H = \Delta P - \Delta C - TC$ gives the return on the hedge :

$$HR = \frac{1}{2}\hat{C}_{SS}S^2\left[\sigma^2\Delta t - \left(\frac{\Delta S}{S}\right)^2 + k\left[E\left|\frac{\Delta S}{S}\right| - \left|\frac{\Delta S}{S}\right|\right]\right] + O\left(\Delta t^{3/2}\right)$$

Where $E(HR) = O\left(\Delta t^{3/2}\right) \to 0$ and $\operatorname{var}(HR) = O\left(\Delta t^2\right)$ both uncorrelated with the market. Thus by integrating this new volatility into the B&S model, a new pricing is done where the option's price depends on the level of the transaction cost. This new price *(now higher)* will give an expected return of zero if used to dynamically hedge. The difference between the adjusted option and the B&S option (Z) could be seen as an insurance policy guaranteeing coverage of transaction costs, whatever those may actually be.

For small transaction costs, $Z = k\,S_0\,N'(d_1)\sqrt{t^*} / \sqrt{2\pi\Delta t}$ and turnover is equal to : Z/kS_0.

Turnover will be greatest when the option is slightly in the money. Testing of the model showed that transaction costs stayed bounded. Even options with 5 years to maturity and $k = 4\%$ gave transaction costs of 10% higher than B&S with turnover of 16.01%.

This model gave a first insight to hedge returns under transaction costs. Testing was done on options with maturity of 3, 6 and 12 months, moneyness (S/X) running from 0.8 to 1.2, revision times every one, fours and eight weeks, and transaction costs of 0, 0.25, 1 and 4%. Even with the close to zero percent hedge return, we would like to mention that variances on those returns were particularly big. For example, hedging a 3 months option with revisions of 8 weeks periods under 4% transaction costs gave mean return of – 4.3% but a volatility of 110.2%. This is in contradiction with the initial assumption that the portfolio is risk free. We can see that the hedge is risky to a certain point and thus should earn a premium.

The authors also tested their new model in comparison to B&S inclusive and exclusive of transaction costs. The results showed a distinct improvement of the hedge return (closer to 0%) but did not reduce their volatility.

Leland's model was the first complete model in the presence of transaction cost. However many questions were left without answers. As mentioned earlier, this model eluted that discrete hedging is feasible; it did not give an optimal answer as to when one should hedge.

The Boyle & Vorst model (1992)

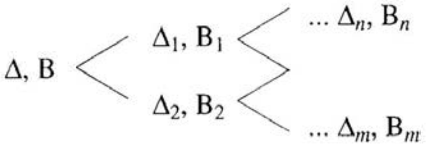
This model was constructed using the same assumptions as the Leland model but instead of a continuous time modeling, it uses the discrete-time binomial lattice framework employed by Cox, Ross, and Rubinstein (1979) for the asset price. Introducing a cost of k to the model enables to rewrite the up and down movements as:

$$\Delta S\bar{u} + B(1+r) = \Delta_1 S\bar{u} + B_1 \tag{4}$$

$$\Delta S\bar{d} + B(1+r) = \Delta_2 S\bar{d} + B_2 \tag{5}$$

$$\textit{where} \quad \bar{u} = u(1+k) \quad \textit{and} \quad \bar{d} = d(1-k) \tag{6}$$

u and d are a measure in % of an up or down movement :



Using this model, at each time step, one needs to calculate the delta and the value of B (the amount of bonds to buy) in order to keep the total portfolio risk free.

Equation (4) indicates that the value of the portfolio in the up state is exactly enough to buy the appropriate replicating portfolio corresponding to this state and to cover the transaction costs incurred in the rebalancing. The model assumes that the replicating institution does not have to buy the initial amount of risky asset (Δ). The value of the call can thus be derived from (4), (5) and (6) by working down the lattice :

$$C = \Delta S + B = \frac{\bar{p}\left[(1+k)\Delta_1 Su + B_1\right] + (1-\bar{p})\left[(1-k)\Delta_2 Sd + B_2\right]}{(1+r)} \tag{7}$$

This gives rise to an adjusted process that differs from both the original asset price and the risk neutral processes. Under this new process, the probability of a particular state depends on whether the previous jump was upward or downward. After a down-jump, the probability of another down-jump is larger than in case of a preceding up-jump. We can represent this conditional probability to be a Markov process with two states:

$$\overline{P} = \begin{pmatrix} \overline{p}_u & \overline{p}_d \\ 1-\overline{p}_u & 1-\overline{p}_d \end{pmatrix} \tag{8}$$

The first column of $\overline{P}$ represents the probability distribution of X_{j+1} if $X_j = log_e\, u$ and the second column represents the probability distribution if $X_j = log_e\, d$. This new process computes the value of a long (short) call as equal to the B&S value but with a modified variance given by:

$$\sigma^2\left(1 \pm \frac{2k\sqrt{n}}{\sigma\sqrt{t^*}}\right)$$

where n is the number of steps until the maturity of the option at T.

This is very close to Leland's valuation of the volatility in continuous time:

$$\sigma^2\left[1 \pm \frac{\sqrt{(2/\pi)}\,k}{\sigma\sqrt{\Delta t}}\right].$$

Since $\sqrt{(2/\pi)} \approx 0.8$ the Boyle and Vorst model will lead to higher (lower) prices for long (short) calls.

The model thus shows that the long call price can be expressed as a discounted expectation under a new Markov process but assumes that the frequency of transaction is specified exogenously.

Tests were done empirically in comparison to B&S and the Leland model. Testing was done on options with maturity of one year, moneyness (*S/X*) running from 0.8 to 1.2, revision times of 6, 13, 52 and 250 times and transaction costs of 0, 0.125, 0.5 and 2%.

The Boyle and Vorst model showed as expected higher long call prices and lower short call prices. Because of this spread between buyers and sellers, market makers should neutralize their positions by entering into offsetting position or charge a higher volatility for selling calls and lower volatility for buying them.

The Boyle and Vorst model was the first complete model in the presence of transaction cost to be implemented using a binomial tree. The model is the discrete equivalent to Leland's model and gave also very similar solutions. Still, this model did not give an optimal answer as to when one should hedge.

The Hoggard, Whalley & Wilmott model (1994)

This model was derived in discrete time to incorporate in the partial differential equation a cost of transaction term. The assumptions are thus the same as in the original B&S except for the fourth and fifth constraints. After a timestep the change in the value of the hedged portfolio is now given by :

$$\delta \Pi = \sigma S\left(\frac{\delta V}{\delta S} - \Delta\right)\phi\, \delta t^{1/2} + \left(\tfrac{1}{2}\sigma^2 S^2 \frac{\delta^2 V}{\delta S^2}\phi^2 + \mu S \frac{\delta V}{\delta t} + \frac{\delta V}{\delta t} - \mu\, \Delta S\right)\delta t - \kappa S |v| .$$

The same as B&S but with the subtraction of the transaction cost. Dynamically hedging the portfolio and choosing $\Delta = \delta V / \delta S$ and v equal to the number of shares to transact will yield a riskless portfolio that should earn the same return as the risk-free return r leading to the new PDF :

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \kappa \sigma S^2 \sqrt{\frac{2}{\pi \delta t}}\left|\frac{\partial^2 V}{\partial S^2}\right| + rS\frac{\partial V}{\partial S} - rV = 0.$$

This equation is a nonlinear parabolic partial differential equation in comparison to B&S's linear PDF. This is very important because in the presence of transaction costs, the value of a portfolio is not the same as the sum of the values of the individual components.

Knowing that Γ for a plain vanilla option is always > 0, integrating the second and the third terms in the previous equation gives a modified variance equal to :

$$\hat{\sigma}^2 = \sigma^2 \pm 2\kappa\sigma\sqrt{\frac{2}{\pi \delta t}} \quad \text{(The same as the Leland equation).}$$

This equation implies that a long position with costs incorporated has an apparent volatility that is less than the actual volatility. When the underlying rises, the owner must sell some quantity of the underlying but because of the bid-ask spread, the price at which it is sold will be lower. The converse is also true for a short position.

The model could be also adjusted for more complex transaction structures :

If trading in shares costs $k_1 + k_2 v + k_3 vS$, where k_1, k_2 and k_3 are a fixed portion, proportional to volume and proportional to value traded respectively, the new PDF is :

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \frac{k_1}{\delta t} - \sqrt{\frac{2}{\pi \delta t}}\,\sigma S\left(k_2 + k_3 S\right)\left|\frac{\partial^2 V}{\partial S^2}\right|$$

$$+ rS\frac{\partial V}{\partial S} - rV = 0.$$

The HWW model was the first model to give a B&S partial derivative solution to value the option including transaction costs. Also, it was the first model to include fixed cost structure in determining optimal hedging. However, the decision as to when to hedge was left arbitrarily to the user.

The Whalley & Wilmott (1993), and Henrotte models (1993)

All the previous models were local in time where the exogenous trading rule was the time of rebalancing. In the two models of this section, the exogenous rule is the risk taken and the timing is flexible.

In these models, the strategy revolves around a bandwidth that runs around the delta of the hedge. If the bandwidth is breach, complete rehedging has to take place. The authors modeled the risk of not being completely hedged after a time step by the following formula:

$$\sigma^2 S^2 \left(D - \frac{\partial V}{\partial S}\right)^2 \partial t \quad ,$$

where D is the actual delta of the position.

If we define H_0 as the maximum tolerable risk for the portfolio then :

$$\sigma S\left|D-\frac{\partial V}{\partial S}\right|\le H_0$$

We can see in the above equation that the left term is the equivalent of the dollar value of the risk of not being completely hedged. The right hand term is the input value that the user has as the maximum tolerable dollar risk that he or she can take during the hedging program. The value is arbitrary and depends on the risk aversion of the user. Thus, rebalancing the portfolio should follow if the bandwidth is breached and the rebalancing should lead to :

$$D-\frac{\partial V}{\partial S}=0\cdot$$

If trading in shares costs $k_1 + k_2 v + k_3 vS$, where k_1, k_2 and k_3 are a fixed portion, proportional to volume and proportional to value traded respectively, the new PDF is :

$$\frac{\partial V}{\partial t}+\tfrac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV$$
$$=\frac{\sigma^2 S^4\Gamma^2}{H_0}\left(k_1+\left(k_2+k_3S\right)\frac{H_0^{1/2}}{S}\right).$$

This model was the extension of the previous HWW model but for the first time, the optimal rehedging decision was based on the output and not arbitrary. This was the first model that actually would tell you when to rehedge the position while taking into consideration the user's risk characteristics. One weak point is the myopic effect that the model has. The decision is made at time t without measuring the effect of the decision on the final wealth at the end of the hedging program.

☐ Global in Time

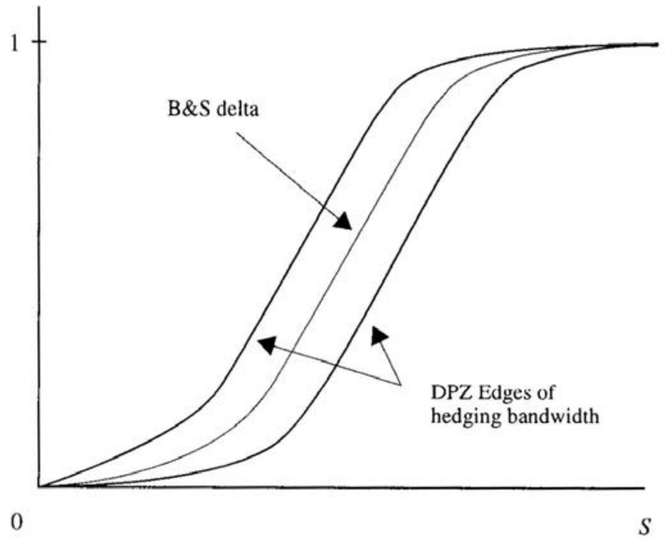
The Hodges, Neuberger (1989) and Davis, Panas & Zariphopoulou models (1993)

As mentioned earlier, the global in time strategy tries to find an optimum without exogenous entries. Ground braking work was first proposed on this subject by HN and later improved by DPZ. The whole idea is to remove any necessary constraints from the derivation as to find a more reliable tool for hedging. HN assume that a financial agent holds a portfolio that is already optimal in some sense but then has the opportunity to issue an option and hedge the risk using the underlying. Because of the rehedging costs, the agent must maximize his expected utility in term of a loss function. Under the HN model, the utility function follows an exponential form having a constant risk aversion and with purpose of valuing an option on its own. The DPZ model improved the solution by including a portfolio effect and constraints on expiry of the option and the obligation due to the option contract. They also assumed that transaction costs are proportional to the value of the transaction (*kvS*). From these assessments, in both the HN and DPZ provide hedging strategies in term of the solution of a three-dimensional free boundary problem *(explained later)*. The variables are S, t, and D. Figure 1 depicts the delta bandwidth prescribed by the utility maximizing model.

A description of the model:

Given an initial cash wealth of $B_{wo}(t)$, the investor can invest in a portfolio of the risky share and riskless bond in an effort to

FIGURE 1



maximize utility. Thus the final wealth expected utility could be written by:

$$J^w\left(t,\ S,\ B_w,\ y_w=0\right)=\max_{yw(\tau),t\le\tau\le T}\left\{E\left|U\left(W_w(T)\right)\right|\right\}\cdot$$

Where the final wealth is the wealth in cash after transaction costs

$$W_w(T)=y_w(S,T)S+B_w(T)-k\left(S,y_w(S,T)\right)+V_T(S)\cdot$$

If we consider now the same investor looking into entering an option position.

The amount the investor is willing to pay to enter the market when they do not have the option position equals the amount they are willing to pay to enter the market when they have an option position plus the value of the option position to them or :

$$\hat{B}_{wo}=\hat{B}_w+V\cdot$$

For a general utility function $U(x)$ and for a proportional transaction cost, $k(S,\ y)=k_3S|y|$, the optimal solution solves :

$$\max\left\{J_y-S(1+k_3)J_B,\right. \tag{9}$$

$$-\left(J_y-S(1-k_3)J_B\right), \tag{10}$$

$$\left.J_t+rBJ_B+\mu SJ_S+\frac{\sigma^2S^2}{2}J_{SS}\ \right\}=0, \tag{11}$$

$$where\quad dB=rBdt-Sdy-k(S,dy)\cdot$$

The (S,y) space divides up into three regions, within each of which one of the terms (9), (10) and (11) equals zero. If (9) equals zero, one would buy the underlying; if (10) equals zero, one would sell the underlying; if (11) equals zero, one does not have to transact.

In their derivation, HN and DPZ used the negative exponential utility function. This function is easy to use since it has a constant absolute risk aversion: $-U''/U'=\gamma$.

$U(x)=1-\exp(-\gamma x)$.

Choosing this utility function produces independence between the monetary amount invested in the risky asset and total wealth and thus the dependence on B could be eliminated. Thus to find the solution to the problem one has to solve a three-dimensional free boundary problem.

The major disadvantage to this solution is the computation that takes very long time and thus could poorly respond to a volatile market.

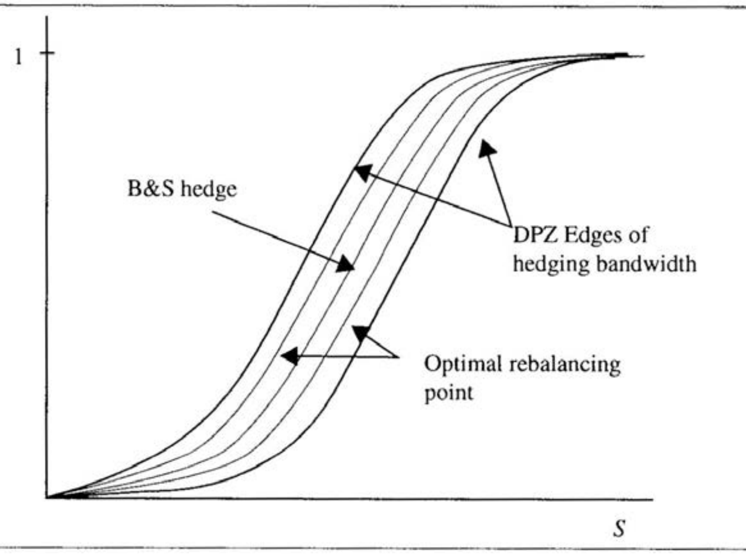
The solution to this utility maximization depends on the transaction cost structure :

If transaction costs are proportional to the value traded, rebalancing occurs if the mishedging hits the hedging bandwidth. The quantity to rebalance is enough to bring the misalignment to the bandwidth.

In case of fixed and proportional costs, upon breaching the bandwidth, rebalancing is ordered to an optimal rebalancing point between the bandwidth and the B&S hedge.

We can see the contrast to Figure 1. In Figure 2, the rebalancing does not need to be done to achieve delta neutrality. Depending on the value of the fixed cost and the variable cost, the model will give a signal as to by which amount to hedge. It could give a signal to completely neutralize the delta bet or to hedge a portion of that risk.

FIGURE 2



The Whalley & Wilmott models (1994, 1997)

As a continuation to the HN and DPZ models, the WW models try to get faster computation of the results through asymptotic analysis of the three-dimensional free boundary problem. Using the same negative exponential utility function as a mean to alleviate the dependence on B, they derived a simple formula. In term of wealth, the utility function could be written as to maximise the final wealth where :

$$J(t,S,B,y)=1-\exp\left(-\frac{\gamma}{\delta(t,T)}\left(B+W(t,S,y)\right)\right).$$

Giving the system of equation to minimize, in the same way as the original HN and DPZ's three regions. The system of equations for J, (9), (10) and (11) transform into :

$$\min\left\{W_y-\frac{\gamma}{\delta(t,T)}\left(S+\frac{\partial k}{\partial y}\right),\right.$$

$$-\left(W_y-\frac{\gamma}{\delta(t,T)}\left(S-\frac{\partial k}{\partial y}\right)\right),$$

$$\left.W_t-rW+\mu SW_S+\frac{\sigma^2S^2}{2}\left(W_{SS}-\frac{\gamma}{\delta(t,T)}\left(W_S\right)^2\right)\right\}=0.$$

If transaction costs are small, WW introduced a parameter called ε as a measure of the size of the transaction costs. At each rehedge there will be an associated cost K that is $O(\varepsilon)$. Then the authors take an asymptotic expansion of the W function in powers of ε. This technique is used to simplify the calculation of the optimal hedge making the problem faster to optimise. To keep the text light, we will not put the different steps of the modification only the final solution :

$$V=\frac{\gamma\sigma^2S^2}{6\delta(t,T)}\left(Y^{+2}+Y^{+}\hat{Y}^{+}+\hat{Y}^{+2}\right).$$

An intuitive way to see this is that after solving for Y^+ and $\hat{Y}^+$, the final value of the option if function of the underlying, time and the option's gamma which is used in the calculation of Y^+ and $\hat{Y}^+$.

This procedure could be used to find the value of the option under different cost structures by changing the above equation. WW give the numerical procedure to evaluate the value under fixed, variable and fixed plus variable structures.

We have to point out the fact that the asymptotic evaluation could only be done for small transaction costs. It would be interesting to see the final result for different utility functions.

New approaches

The Gilster model (1997)

Using options to hedge a portfolio of stocks is in theory a good idea but in practice, the options are very unstable and thus hedging with option is inherently a very complex problem. B&S recognized this and demonstrated using the Taylor series that the hedge return, after eliminating powers in higher order of two, produces returns that are a function of the square of the change in the underlying stock. Thus for short rebalancing intervals, the price changes are uncorrelated with the market. In his model, Gilster argues that comparing the hedge to a perfect position *(when the hedge was last rebalanced)* undervalue the risk of the hedge and allowing for long periods without rebalancing. Why? Just after rebalancing, because of the unstable nature of the option, the portfolio will become unbalanced. Thus the instantaneous risk is equal to the risk of the new perfectly hedge if you would rebalance plus the risk of the unhedged stock you continue to hold because you did not rebalance. We can see this unbalanced portfolio as a combination of a perfectly hedged portion and another that is completely unhedged. Let us consider an option hedge in the form of long stock and short option:

The initial position is equal to

$$H = V_S\{S,t\}S - V\{S,t\}.$$

After a timestep, the hedge is equal to

$$H = V_S\{S,t\}[S+\Delta S] - V\{S+\Delta S, t+\Delta t\}.$$

The dollar value needed to rebalance is

$$[V_S\{S+\Delta S, t+\Delta t\} - V_S\{S,t\}][S+\Delta S].$$

If the portfolio is not rebalanced then $H = P + U$

where $P = V_S\{S+\Delta S, t+\Delta t\}[S+\Delta S] - V\{S+\Delta S, t+\Delta t\}$

and $U = -[V_S\{S+\Delta S, t+\Delta t\} - V_S\{S,t\}][S+\Delta S]$.

Thus the instantaneous risk or standard deviation of the position is : $\sigma_H = \sigma_S \frac{|U|}{H}$,

and the instantaneous beta is : $\beta_H = \beta_S \frac{U}{H}$.

We can see that the unhedged portion is as risky as the underlying. The author shows that volatility could be very high for medium rebalancing times and the hedge is not risk-free as B&S and other articles showed. We saw this phenomenon in the Leland calculation where the hedge return was close to zero but its volatility was really high. This demonstration shows that we have to model the risk of the hedge in a way to pickup intertemporal risk and not assume that this risk is uncorrelated with the market. This phenomenon is even more important in option spreads where ones try to hedge an option with another option.

Gilster's tests were done on out, in, and at-the-money calls with small changes upward in the underlying and options with 3 and 6 months expiration. In cases of stock option hedges, standard deviations reached 42% of the underlying and 93% in case of option spreads. Very much higher than what is supposed to be a riskless portfolio.

CONCLUDING REMARKS

It is widely known that while ignoring time value return, any price movement in either direction causes the hedge to lose money. If the hedge is rebalanced continuously, the loses stemming from $\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}$ will be offset on average by gains on time value.

This is only possible for short rebalancing times otherwise the losses from gamma will overwhelm the modest gains from theta. In our view, this hedge resembles a short straddle *(short one put and a call with the same strike price)*. This is the same as having large positive beta when the market declines and large negative beta when the market rises. To stay afloat or make modest gains, only very small movements in the underlying are permissible otherwise, big loses could be incurred. Viewed this way, the hedge is far from being the riskless portfolio described by B&S. We acknowledge this phenomenon and we will take it into account in the construction of the optimal rehedging pattern so as to keep the portfolio's risk as low as possible.

Even thought many researchers have witnessed the existence of discrete hedging over long periods, some questions are still unresolved. The B&S model has some big gaps when applied discreetly. By construction, the model eliminates higher orders that do contain systematic risk. We also think that existing hedging techniques in the literature are not applicable in reality. They measure the risk from a perfect position at t equal zero concluding wrongly the non-existence of correlation.

Economic and financial markets are driven by many factors including investor psychology. The market's psychology is very hard to understand and do twist the expected pricing. To fully appreciate this let us go back to pre 1987. At that time, it was true that not all strike prices for a given underlying implied the same level of volatility as determined from a standard B&S model. There was a tendency for out-of–the-money options to trade at higher implied volatilities than the at or near-the-money variety. This "smile" effect was noted in the literature and was explained in a variety of ways. Some argued that the real distribution is not lognormal; others said that it follows what economist name, the lottery effect. Then came the crash.

☐ Enter the skew

In one day, the smile disappeared to open the door for today's skew. There was now a consistently observable tilting of the traditional smile. The skew has the effect of giving higher implied volatility to options whose strikes were below the current market fair value. Why did this happen is empirically unknown but we bet that some investors were so badly hurt, being naked with out-of-the-money options that they put now a higher premium for security. We buy life insurance even thought we know that on average it is a loosing bet. We would rather pay a premium to receive that sum of money if we were unfortunate to meet the Grim Reaper. It is the same for options and the previously mentioned fat tail. We pay a premium for those who will bear the risk of another meltdown. Maybe this premium is overdone, but it exists and we have to work with it.

Finally, if discrete risk-free hedging is not possible, meaning that portfolio hedging could not eliminate all or much of the systematic risk, we propose to go back to older models as Boness

(1964) and Sprenkle (1964) ho used a stock's growth rate to price the options. Maybe the B&S model does contain too many unforgiving parameters hindering its generalized use.

■ APPENDIX : NOTATION USED IN THE LITERATURE REVIEW

t	=	Current calendar time
T	=	Expiration time
t^*	=	Time to expiration date or $T - t$
C	=	Price of a European call option at time t based on B&S
$\hat{C}$	=	Price of a European call option at time t based on a new specified model
V	=	Price of a European option a time t
Π	=	$V(S, t) - \Delta S$
PDF	=	Partial Derivative Formula
r	=	Riskless interest rate continuously compounded
μ	=	Expected return
S	=	Stock price at time t
X	=	Exercise price of the option
Δt	=	Rebalancing period
$S + \Delta S$	=	Stock price at time $t + \Delta t$
$C + \Delta C$	=	Call price at time $t + \Delta t$
σ^2	=	Variance rate of return or implied variance
d_1	=	$\left[\log(S/X) + \left(r + \frac{1}{2}\sigma^2\right)T\right] / \sigma\sqrt{T}$
d_2	=	$d_1 - \sigma\sqrt{T}$
$N(\cdot)$	=	The cumulative normal distribution function
$N'(x)$	=	$\frac{1}{\sqrt{2\pi}} e^{-(1/2)x^2}$
$Z(\cdot)$	=	The unit normal density function

Θ = The first derivative of V to time

Δ = The first derivative of V to the underlying

Γ = The second derivative of V to the underlying

ϕ = A value drawn from a standardized Normal distribution

HR = The hedge return

C_x, V_x = Partial derivative of the call value and option value with respect to x

μ_3 = Skewness of the distribution

$E[x]$ = The expected value of x

k = A round trip transaction cost

B = The value invested in bonds

J^{wo} = The maximized expected utility of final wealth without the option

J_x and W_x = Partial derivative of utility and wealth respectively to x

$y(S, t)$ = The quantity of shares to be held in the portfolio

τ = Any time between t and T

γ = $-U''/U'$, γ is a measure of risk aversion of an investor

$\delta(t,T)$ = $e^{-r(T-t)}$, is a discount factor which convert wealth at maturity to current

Y = The difference between the actual and ideal number of shares to hold

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